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Final Report

Development of Constitutive Equations for Nuclear Grade Graphite for Space Applications

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Development of Constitutive Equations for Nuclear Grade Graphite for Space Applications II

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CONTENTS

LIST OF SYMBOLS

A	Applied force, a tensor
a, b, c, e, f, C, M, N	Constants
b	Critical value of a friction block
$\mathfrak{D}, \mathfrak{F}, \mathfrak{G}$	Functionals
E	Strain tensor, scalar strain
E^*	Permanent set
F	Deformation gradient
G	Nonlinear operation
$\{G\}$	Group
g	Weight of a determinant
H, J	Functions
h	Orthogonal function, axial vector
I	Identity
K	Kernel function
k	Spring constant, permanent set constant
L	Linear transformation
ℓ	Length, matrix with unit in the i-i place
m	Slope of the loading curve
R	Rotation tensor
s	Arc length
t	Present time
U	Right stretch tensor
u, v, w	Vectors in V

V	. . . .	n-dimensional vector space
W_i	. . . .	Arbitrary quantities
X, x	. . . .	Positions of the generic particle
z	. . . .	Strain hardening parameter
Δ	. . . .	Determinant
δ	. . . .	Delta function
π	. . . .	Piola-Kirchoff stress tensor
σ	. . . .	Stress
τ	. . . .	Time variable

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ABSTRACT

The object of this investigation was to develop a general constitutive equation for the room temperature mechanical response of nuclear grade graphite. The distinguishing characteristics of this response are (i) the stress-strain relation appears to be independent of the rate of loading short of impact rates; (ii) the response exhibits a pronounced degree of transverse isotropy; and (iii), the material undergoes permanent plastic deformation even at the lowest stress levels.

The greatest portion of this report is devoted to the derivation of constitutive relations for the scalar and three dimensional cases. The scalar theory succeeds in demonstrating rate independence and plasticity while the three dimensional theory incorporates transverse isotropy.

Comparison of the theory is made with some available uniaxial cyclic load-strain data for ATJ graphite. The scalar theory provides an excellent correlation with experiment while the three dimensional theory gives a reasonably good conservative correlation.

I. INTRODUCTION

A. Purpose and Scope of Research

The object of this investigation was to develop a general constitutive equation for the room temperature mechanical behavior of nuclear grade graphite. To this end, this report presents a constitutive theory for the rate independent transversely isotropic response of graphite or any other material exhibiting similar distinguishing characteristics. A specific application of the theory is made to uniaxial cyclic load-strain experiments. A comparison of the theoretical and experimental results is given.

In this analysis the stress tensor is taken to be a tensor functional of some measure of deformation. Invariance requirements place restrictions on the form of these functionals. If a function is considered as a mapping of a set of numbers onto another such set of numbers, then a functional may be considered a mapping of a set of functions onto another such set of functions. Involved in these mappings are kernels, called material functions, describing the behavior of the material. For example, in the case of a linear viscoelastic material which is isotropic and whose mechanical response is described by means of a linear functional, two material functions are needed, say the shear and bulk relaxation functions.

Experimental data for ATJ graphite consisting of stress-strain (longitudinal and transverse) data for uniaxial loading parallel to the three major material axes is available. This data is sufficient to determine certain of the response parameters in special cases, and it is used as a quantitative evaluation of the representation.

It is presumed that graphite is a rate independent material which is transversely isotropic and history dependent. These assumptions are, in fact, verified by experiment. Many physical systems possess the property of having their output dependent only upon the present value of the input. In the majority of cases, however, the output of a system depends in some way upon the past history of the input. For instance, the temperature, at a given instant of time, in an electric furnace is not only dependent on the current flowing in the heating element at that instant, but also on the past history of the electric current applied.

A material is said to be rate independent if the stress at any instant of time depends on the deformation history, but not on the rate at which the deformation history was executed. For example, a linear viscoelastic material will exhibit different stress outputs for the same inputs when these inputs are applied at different rates. Graphite, however, exhibits the same output for all input rates short of impact intensity.

B. Graphite

1. General characteristics

Graphite in its natural form, was doubtless known to prehistoric man and may even have been put to use by some of the ancient civilizations. The first published reference to graphite is found in the Natural History of Ferrante Imperato in 1599 [1]*. Imperato called graphite "graphio piombino." The name graphite was, however, originated by Abraham Gottlob Werner. The scientific investigation of graphite began toward the end of the eighteenth century. In 1799 Karl Wilhelm Scheele discovered that graphite was mineralized coal. Allen and Pepys, showed in 1807, that charcoal, diamond, and graphite left the same residue after they had been burned. Since that time much progress has been made. The crystal structure of graphite has been determined. The thermal, mechanical, and atomic properties of graphite have been investigated. Graphite has been produced artificially and it has lent itself to many applications in industry [2].

Graphite is a form of pure carbon. Along with diamond and charcoal it is one of the three forms of carbon found in nature. The difference between these three forms of carbon is that diamond crystallizes cubically, charcoal crystallizes amorphously, and graphite crystallizes hexagonally [3].

The ideal graphite crystal structure, as shown in Fig. 1 Page 97, can be seen to possess a layered structure. As a consequence

* Numbers in brackets refer to entries in the bibliography.

of the relatively small distances between the carbon atoms in each layer, 1.42A°, strong bonding exists between the atoms in these layers. The bonding between successive layers is much weaker due to the relatively large distances, 3.35A°, between these layers. This results in the easy displacement of the layers relative to each other and accounts for the fact that graphite is often used as a lubricant.

Other consequences of this layered structure are evidenced by the pronounced anisotropy of many of the physical properties exhibited by graphite [4]. The various anisotropy ratios of graphite are now discussed in terms of the ratio of the "weak" axis to the "strong" axis. In its "strong" direction graphite is probably harder than diamond. For an ideal single crystal the anisotropy ratios of hardness may be as low as 1/100 or even 1/1000. Graphite is extremely compressible normal to the network planes and anisotropy ratios have been estimated to be on the order of 10^4 or 10^5. It should be noted that polycrystalline graphite consists of an agglomeration of small crystals at various orientations, with, on the average, 20% free space or porosity.

Aside from its anisotropic properties graphite is colorless, tasteless, non-toxic, and almost chemically inert. Graphite has a very low coefficient of thermal expansion along with the fifth highest thermal conductivity of any material and it is a good conductor of electricity. Unlike most materials the strength of graphite will increase with temperature to at least 2500°C. Graphite has one of the highest strengths per unit weight of any material, and at temperatures above 1600°C it is superior to any known metal or ceramic.

2. Graphite as a Material with Memory

Most materials have mechanical properties which are in some manner, dependent upon the past history of some mechanical variable [5]. A material of this type is commonly described as a material with memory. If this material exhibits some mechanical property which is influenced to a great extent by events which have occurred in the recent past, and is influenced to a lesser extent by events which have occurred in the more distant past, then this type of material is termed a material with fading memory. Various types of materials are observed to exhibit various degrees of fading memory, and indeed, materials with virtually no memory are also common.

An example of a material which lacks memory effects is an elastic material. The present mechanical state of an elastic material is not dependent upon its history of deformation, but only upon its present mechanical state. Thus, an elastic material has mechanical properties independent of its deformation history. A viscoelastic material often has a fading memory of deformation, since deformations which have occurred in the far distant past might have little influence on the present mechanical state of the material, as compared with deformations which have occurred in the recent past and which have great influence on the present mechanical state of the material.

In subsequent work presented in this dissertation, functional relations which relate the present state of stress in a material to its history of deformation will be dealt with. Functional relations of this type will be derived which, when applied to graphite, will be capable of describing certain mechanical properties of the material. At room temperature, graphite does not exhibit the fading memory property. In fact, graphite might be classified as a material with perfect memory. Events, i.e. deformation histories, which have occurred long in the past may have as much influence on the present state of stress as events which have recently taken place.

The physical processes taking place within graphite, which are generally thought to be responsible for this perfect memory property, are classified as plastic yielding phenomena [6], [7], [8]. Plastic yielding occurs in graphite at even the smallest strains contributing to the yielding of the material, and the effect of these strains will be felt for all time. This is the justification for classifying graphite as a material with perfect memory. It should also be noted that room temperature graphite, unlike materials which exhibit fading memory for histories which do not yield the material and are classified as elastic-plastic or elastic-viscoplastic, never exhibits the fading memory property.

3. Qualitative Mechanical Response of Room Temperature Graphite

The qualitative characteristics of the uniaxial stress-strain relation for room temperature graphite as shown in Fig. 2 and as described in [10], [11], [12], [13], [14] are now described.

It is evident, from Fig. 2, that the stress is some monotonic increasing continuous function of the strain so long as the load is applied monotonically. This implies that along the curve C,

$$\pi = f(E), \tag{1.1}$$

where π is the stress, ε is the strain, and f is a monotone continuous function of its argument. If the specimen under consideration experiences a reversal of the applied load at point A on the path C, the material will exhibit a permanent set $\varepsilon^*(A)$ when the load has been completely removed. The permanent set is a function only of the maximum strain achieved prior to the unloading,

$$\varepsilon^*(A) = g(\varepsilon(A)) \tag{1.2}$$

where g is a function of its indicated argument. Implicit in (1.2) is the assumption that the permanent set $\varepsilon^*(A)$ is dependent only upon the strain at the point of unloading. Similarly, for some other point B on C,

$$\varepsilon^*(B) = g(\varepsilon(B)) \tag{1.3}$$

An indication of the dependence of the permanent set on the strain at the point of unloading will be discussed later.

If the specimen of graphite is reloaded at the point with coordinates $(\varepsilon^*(A), 0)$ (see Fig. 2) the reloading path will be different from the loading path. The important thing to observe in this case is that the reloading curve has a shape different than that of the unloading curve, and these paths do not coincide as they do for an elastic-plastic material. Thus, we have a hysteresis loop formed by the unloading-reloading process.

Consider a specimen of graphite in its undeformed state. Then if it is loaded and

$$\frac{dE}{dt} > 0 \tag{1.4}$$

for the entire process, and fracture occurs at the point F, where $E(F)$ is the value of the strain at fracture, the path C will be followed until the strain $E(F)$ is attained. The form of the one dimensional stress-strain relation to the point of fracture will be derived later on in our analysis.

Let the material be unloaded at some point A on C, where

$$0 < E(A) < E(F). \tag{1.5}$$

This will result in a nonlinear unloading path. Unloading at some other point B on C,

$$E(A) < E(B) < E(F) \tag{1.6}$$

will result in another unloading path which will not, in general, be parallel to the unloading path from point A on C, and

$$E^*(B) > E^*(A) \tag{1.7}$$

Reloading at $(E^*(A), 0)$ will result, in general, in a nonlinear path which may not pass through the point A on C, but will intersect C at some small distance, d, to the right of A.

If, however, upon reloading from $(E^*(B), 0)$, the previous maximum strain $E(B)$ is not attained, but the strain $E(H)$ is reached, then the unloading path from H to $E^*(B)$, as shown in Fig. 2, is followed. A most interesting property of graphite is that it behaves like an elastic material insofar as there is no further change in the permanent set for strains which do not exceed a previous maximum strain. In fact, if we let a specimen of graphite be loaded to $E(A)$, unloaded to $E^*(A)$, and then loaded and unloaded to these two points a total of n-times, the loop bounded by $E^*(A)$ and $E(A)$ will then be traced out n-times. For this reason graphite may be considered an elastic material in the above sense.

Of the mechanical phenomena observed in graphite one of the most significant is its rate independent behavior. All uniaxial loading programs, short of impact intensity, which achieve a given fixed strain will produce the same stress-strain curve C. Consequently graphite is classified as a rate independent material. The condition of rate independence automatically rules out the possibility of observing any time dependent phenomena such as creep or stress relaxation in graphite. Consequently, and without any loss in generality, a convenient uniaxial loading program may be chosen to represent any actual uniaxial loading program.

II. MECHANICAL MODELS

In an attempt to better understand the mechanical properties of room temperature reactor grade graphite, several mechanical models will be constructed. These models are presented here in order to obtain an intuitive understanding of the mechanical properties of graphite. The models which will be described below have various advantages and disadvantages. A disadvantage common to all these models is that they are one dimensional. Three dimensional relations can be developed within the framework of the incremental theory of plasticity which reduce to the equations obtained for the one dimensional mechanical models - an interesting method of doing this is given by Iwan [15]. Three dimensional models have the disadvantages of not appealing to our intuition, being difficult to construct and yielding unweildy equations. The models which follow are similar to one another in that they may be constructed using only two elements, the spring element and the friction element [9], [16], [17].

The spring element is defined as that element whose deformation, at any time, is a function of the total applied force. This element is pictorally represented in Fig. 3. The applied force is denoted by A, and ℓ is the length of the spring at the present time. Let ℓ' be the length of the spring initially, or the length of the spring when $A=0$. Our definition requires that,

$$A = f(E) \tag{2.1}$$

where $E = \ell - \ell'$ is the elongation.

For the special case when (2.1) represents the applied force as a linear functions of the elongation, the spring element becomes the usual Hookean element. The following expression describes the Hookean element,

$$A = kE \tag{2.2}$$

where k is called the spring constant.

The friction element, or friction block, is pictorally represented in Fig. 4. The friction block is sometimes called the St. Venant element. If the magnitude of the applied force A is less than the maximum possible force between the block and the plane it rests upon (the critical value b), then the applied force will not be of sufficient magnitude to move the block. This may be expressed by means of the following relations,

$$\begin{cases} \text{if } |A| < b, & \text{then } \dot{E} = 0 \\ \text{if } A \geq b, & \text{then } \dot{E} > 0 \\ \text{if } A \leq -b, & \text{then } \dot{E} < 0 \end{cases} \tag{2.3}$$

The following model has been constructed in order to give a qualitative description of the hysteresis loop exhibited by a graphite specimen when continuously loaded in tension and compression. The model consists of three elements, two elements being Hookean and the other a friction block. This is illustrated in Fig. 5.

Let E_1 be the deformation of the spring with constant k_1, and let E_2 be the deformation of the spring with constant k_2. The total deformation will be E, where

$$E = E_1 + E_2 \tag{2.4}$$

The mathematical description of this deformation is given by the relation

$$A = k_1 (E - E_2). \tag{2.5}$$

The deformation E_2 may be determined by integrating $\dot{E}_2$. $\dot{E}_2$ is given by the following description,

$$\begin{cases} A - k_2 E_2 = b & \text{and } \dot{A} \geq 0 \quad \text{imply } k_2 \dot{E}_2 = \dot{A} \\ A - k_2 E_2 = b & \text{and } \dot{A} < 0 \quad \text{imply } \dot{E}_2 = 0 \\ A - k_2 E_2 = -b & \text{and } \dot{A} \leq 0 \quad \text{imply } k_2 \dot{E}_2 = \dot{A} \\ A - k_2 E_2 = -b & \text{and } \dot{A} > 0 \quad \text{imply } \dot{E}_2 = 0 \\ |A - k_2 E_2| < b & \qquad\qquad \text{implies } \dot{E}_2 = 0 \end{cases} \tag{2.6}$$

These properties are represented by the stress-strain diagram in Fig. 6. It can be seen from Fig. 6 that when the graphite sample is loaded in tension the stress-strain curve will follow the path OF as the spring with constant k, deforms and the lower element remains stationary. The instant point F is reached the critical value b is attained and the lower element of the model will begin to move, tracing out the path FB. If, however, the applied load is removed at point B, the lower element will once again become stationary and preserve its maximum deflection $E_2(B)$. It is, of course, assumed that the force of the spring with constant k_2 is less than b. The path BC will then be traced as the spring with constant k_1 returns to the unstressed state. At this point, the total displacement is the displacement in the lower element. If the material is reloaded, the path BC is again followed and continues along BD.

If, instead of assuming the spring to be linear, it is assumed that the general relation (2.1) holds and the general relations corresponding to (2.3) are obtained, a stress-strain curve analogous to the one shown in Fig. 6 is obtained. This curve is shown in Fig. 7. The interpretation of this behavior is analogous to that given above for the linear model. This model is a better description of the properties of graphite than the linear model.

It can be seen that this model has the advantage of not allowing for any time dependent properties in the material. Thus, the material represented by Fig. 6 or Fig. 7 will not exhibit the properties of creep, stress relaxation, or dependence upon the rate of loading. A major drawback of this model is that while it may be capable of quantitatively duplicating a given stress-strain curve for graphite by defining the appropriate function in (2.1), it does not predict a permanent set for any applied stress less than b. Thus, one of the most distinguishing properties exhibited by graphite is beyond the descriptive powers of this simple model.

A model proposed by Jenkins [6] is now constructed. Jenkins observed that the stress-strain curve for polycrystalline reactor grade graphite at room temperature is parabolic for small strains. He then constructed a model which yielded a parabolic law. This theory did not, however, predict a method for determining the value of the quadratic coefficient.

Jenkin's model is based on the assumption that when graphite is subjected to stress cycling under low compressive stresses the applied stress is large enough to only produce plastic deformations in just a few isolated parts of the structure. The mechanism of deformation is assumed to be plastic yielding. This plastic deformation is limited by a restraining elastic matrix. The parts of the material undergoing plastic deformation are imbedded in the restraining matrix and these parts cease to deform as soon as the applied stress within each matrix is decreased below the yield stress b , of these areas.

The mechanical model for this type of deformation is again made up of a series of friction blocks and spring elements. Here, However, each block is backed by a spring, as shown in Fig. 8. Here the block will move only when the applied force A exceeds the frictional force b . The motion of the block will then cease when the elastic reaction in the backing spring is built up until it reaches $A-b$. The generalization based upon the above assumption is an extension of the model in Fig. 8. It is a series of equal friction blocks alternating with equal backing springs. As can be seen in Fig. 9, as the applied force A increases more blocks begin to move with each block building up a back stress in its backing spring. If the applied force is removed, the first element will relax only when the stress in the backing spring can overcome the friction force of the block.

Applying this, Jenkins obtains the relation,

$$E = k_1 A + k_2 A^2 \tag{2.7}$$

where $k_1 = \frac{1}{E}$ is the inverse of Young's modulus at infinitesimally small strains. The quadratic coefficient, k_2, cannot be quantitatively determined from Jenkins' theory. Jenkins also presents an equation which describes the unloading path for small strains in terms of a quadratic law.

Woolley [8] has obtained a mathematical representation which predicts, accurately, the loading path of graphite. Whereas Jenkins' model is valid only for small strains (up to 0.25%) Woolley's model provides a good fit for all values of strain and predicts a finite compressive strength. Woolley, however does not attempt description of the unloading curve.

Woolley assumes that a given specimen of graphite will contain N_o dislocations distributed throughout its interior when a given stress is applied. As in the previous mechanical model, each dislocation can be thought of as being represented by a friction block. Since the movement of each dislocation is limited by a restraining elastic matrix each friction block will be backed by a Hookean element. Let the yield stress of each dislocation be b. It then follows that a given dislocation will remain stationary when the force on that dislocation is less than b. When the force on the dislocation is greater than b each dislocation will move a distance ℓ. Taking the average over all the dislocations in the specimen by defining an appropriate distribution function, Woolley obtains the relation,

$$A = YE_0 \left[1 - \exp(-E/E_0) \right] \qquad (2.8)$$

The constants Y and E_0 depend on the elastic moduli of the stress-strain relation for graphite, the method preparation of the graphite specimen and its degree of preferred orientation.

III. MATHEMATICAL PRELIMINARIES

A. Constitutive Functionals

Let a graphite specimen G occupy a specific region in a three dimensional Euclidean point space. We may consider G to be made up of elements, which are called particles of G. Let x be the position vector of a generic particle $\mathcal{X}$ of G in Euclidean space. Now if G is deformed in an arbitrary manner, then at some time τ the generic particle $\mathcal{X}$ will be at the place which has the position vector x. The motion of the generic particle from $\mathcal{X}$ to x is denoted by,

$$x = \chi(\mathcal{X}, \tau). \tag{3.1}$$

Thus, the function χ defines the deformation process. Assume that at the present time t, the state of stress $\sigma(t)$ at a material point is a function not only of the deformation gradients at time t, but also a function of the values of the deformation gradients at all times prior to t. Here the deformation gradient $F(\mathcal{X}, \tau)$ is defined as the gradient of $\chi(\mathcal{X}, \tau)$ and is a second order tensor. In coordinate notation the deformation gradient at time t may be written as,

$$F_{ij} = x_{i,j}(\mathcal{X}, t) = \frac{\partial x_i}{\partial \mathcal{X}_j} \tag{3.2}$$

Thus, it is assumed that the material response is dependent upon the entire history of its deformation gradients and a material of this type is called a history dependent, or a memory dependent material [18], [19].

The present state of stress in graphite is expressed as a function whose value at any point X at time t is explicitly expressed as the result of some operation upon the infinite set of values assumed by the deformation gradients over some continuous function of time. Such an expression is termed a functional. Hence the present state of stress in our material may be written as a functional of the deformation gradients F over the time interval $-\infty < \tau \leq t$. Using the notation invented by V. Volterra [20], this is indicated by

$$\sigma(t) = \mathfrak{F}\left[\underset{\tau=-\infty}{\overset{\tau=t}{F(\tau)}}\right]. \tag{3.3}$$

Let us assume that our graphite specimen has been physically standardized for use at some time, say $\tau = 0$. In our case this might correspond to the time at which the specimen of pryolitic graphite was removed from its oven. This allows us to rewrite the constitutive equation (3.3) as,

$$\sigma(t) = \mathfrak{F}\left[\underset{\tau=0}{\overset{\tau=t}{F(\tau)}}\right] \tag{3.4}$$

The principle of objectivity requires that all constitutive relations be independent of the observer. Application of this principle allows (3.4) to be rewritten in the form [18],

$$\pi(t) = R^{+}(t)\sigma(t)R(t) = \mathfrak{F}\left[\underset{\tau=0}{\overset{\tau=t}{E(\tau)}}\right] \tag{3.5}$$

where $\pi(t)$ is the Piola-Kirchoff or rotated stress tensor. R is the rotation tensor obtained by a polar decomposition of F. The polar decomposition theorem states that $F = RU$ where U, the right stretch tensor, is positive definite and symmetric and the rotation tensor R is orthogonal. E in (3.5) is defined by,

$$E(\tau) = \tfrac{1}{2}(U^2(\tau) - I) = \tfrac{1}{2}(F^+(\tau)F(\tau) - I). \tag{3.6}$$

U^2 is called the right Cauchy-Green tensor and E is the strain tensor. The above restriction due to objectivity, replaces arbitrary functional dependence upon the nine components of F by arbitrary functional dependence upon the six components of E, since E is symmetric.

The constitutive relation (3.3) can be taken as the definition of a simple material. The assumption that a material is simple is an assumption of a very general nature. Indeed, most material theories are subsumed by the theory of simple materials. For example, the theories of linear and nonlinear viscoelasticity, the theory of dislocations and various special theories are derivable within the framework of the theory of simple materials.

B. Rate Independence

Thus far the material has been allowed to be dependent upon its rate of deformation. By observing the properties of graphite one can conclude that while the stress may be dependent upon the deformation gradients it is not dependent upon the rate at which the deformation is executed. This is analogous to the theories of classical elasticity and plasticity where the rate of deformation does not influence the stress. If the assumption of rate independence is applied to the constitutive relation (3.5) certain simplifications result.

In order to make the hypotehsis of rate independence explicit in the constitutive relation (3.5) the strain history $E(\tau), \tau \geq 0$ must be specified in terms of its path in E -space and the rate of traversal of this path. E -space is defined as the space formed by the components of the strain tensor.

Following the theory of rate independent material as developed by Pipkin and Rivlin [21], the arc length $s(\tau)$ which has been traversed up to time t may be defined by,

$$s(\tau) = \int_0^{\tau} \left[\frac{dE(\tau')}{d\tau'} \cdot \frac{dE(\tau')}{d\tau'} \right]^{1/2} d\tau' . \tag{3.7}$$

This function increases monotonically for all admissible inputs $E(\tau)$. The field path in E -space is described by giving the dependence of E parametrically upon the arc length. It has been assumed that the material was physically standardized at $\tau = 0$, thus all paths begin at zero strain, i.e. $E(0) = 0$, and $s = \mathfrak{s}$ when $\tau = t$.

A rate independent material described by (3.5) may be written as

$$\mathfrak{F}[E(\tau)] = \mathfrak{F}[E(s)] \tag{3.8}$$

for all transformations

$$s = s(\tau) \tag{3.9}$$

where $\mathfrak{s}$ increases monotonically in time. The function (3.7) is obviously a time invariant function. Thus we may write,

$$\pi(\mathfrak{S}) = \mathfrak{F}\left[E(S)\right]_{S=0}^{S=\mathfrak{S}} \tag{3.10}$$

If $E(S)$ is differentiable at each point on the strain path and $E(0)=0$, then (3.10) can be written as

$$\pi(\mathfrak{S}) = \mathfrak{D}\left[\frac{dE(S)}{dS}\right]_{S=0}^{S=\mathfrak{S}} = \mathfrak{D}\left[\dot{E}(S)\right]_{S=0}^{S=\mathfrak{S}}. \tag{3.11}$$

Due to the fact that a rate independent material does not exhibit any explicit dependence upon time, it is intuitively clear that the constitutive relation (3.11) will not allow for any time dependent properties such as ageing effects, creep recovery, or stress relaxation.

It is easy to show the validity of the above statement. Consider for the moment an arbitrary rate independent material whose constitutive equation can be either (3.10) or (3.11) in as much as (3.10) and (3.11) are equivalent. Now a material obeying (3.10) is defined to lack ageing effects if the mechanical properties of the material in its undeformed state do not change in time. This implies that if ageing effects are present they must take the form of chemical or structural changes in the material, since by (3.7) $S(\mathcal{T})$ will be zero for all times at which no strain is applied. Thus, there can be no change in the state of stress in the undeformed material since there is no input to the material in its undeformed state. This means that the constitutive equation (3.10) will not allow for ageing effects, because in (3.10) a non-zero input is required to produce a non-zero output.

A material is said to lack the property of stress relaxation if after the removal of an applied strain an instantaneous stress recovery is now followed by a gradual stress recovery. We will show here that there can be no stress relaxation at constant applied strain. It is obvious from (3.7) that if the strain input is constant the change in the strain with respect to the time variable will vanish and the arc length will remain constant. That is, since $\frac{dE}{d\tau}$ vanishes for any constant strain input, there can be no further increase in the arc length. Thus, if the material is loaded to a fixed value of strain and then, at time t_0, held at that fixed value until t, the equality

$$\mathfrak{S}\left[E(s)\right]_{s=0}^{s=s_0} = \mathfrak{S}\left[E(s)\right]_{s=0}^{s=\$} \tag{3.12}$$

must be satisfied for all $\$ = s_0$ provided that no further strain is applied. Thus, it has been shown that any rate independent material will not be able to exhibit the property of stress relaxation under constant strain.

A material is said to lack the creep recovery property if upon removal of an applied stress an instantaneous strain recovery is not followed by a gradual strain recovery. In order to discuss the phenomenon of creep recovery it is necessary to have a stress input. Let us, therefore, assume that the constitutive relation (3.10) may be inverted. In this case we may write (3.10) in the form,

$$E(\$) = \mathfrak{G}\left[\pi(s)\right]. \tag{3.13}$$

The arc length must now be redefined if the stress is to be the input. Since the arc length is defined on the input space it must, in this case, be defined on the stress space in order that (3.13) be consistent with (3.10). Thus the arc length is defined by,

$$s(\tau) = \int_0^{\tau} \left[\frac{d\pi(\tau')}{d\tau'} \cdot \frac{d\pi(\tau')}{d\tau'} \right]^{\frac{1}{2}} d\tau' \qquad (3.14)$$

where as in the previous case an inner product operation is indicated. With this definition and the constitutive relation (3.13) it can be shown that the material will not exhibit any creep recovery properties. All that need be done is to follow verbatim the discussion on stress relaxation.

C. Functional Approximations

The start of the twentieth century saw certain investigations made by various French mathematicians into the nature of functionals. The most prominent of these were M. Frechet, his principal advisor J. Hadamard, and R. Gateaux. They showed that under certain conditions a functional could be represented as a sum of a series of multiple integrals. Frechet [22], [23], in particular, showed that if $\mathcal{U}$ is a linear functional defined on a set of functions which has the property that if the functional $\mathcal{U}(f_n)$ converges to $\mathcal{U}(f)$ whenever f_n converges to f uniformly, then $\mathcal{U}$ can be represented by a Fourier series.

V. Volterra [24] used a representation similar to Frechet's and showed how history dependent phenomena, represented by functionals, give rise to nonlinear constitutive equations which may be represented as sums of multiple integrals. Volterra then applied his results to history dependent physical processes in elasticity, electromagnetism, and other areas of physics.

Green, Rivlin, and Spencer [19], [25], [26], presented a fairly rigorous treatment of three dimensional constitutive equations for materials with memory. They assumed that the stress tensor was dependent upon the entire history of the displacement gradients. They then represented the constitutive functional, using Frechet's theory, by means of a sum of multiple integrals of the deformation history with certain material functions as kernels. It was also assumed that the material was at rest before time $t = 0$, requiring the domain of the constitutive functional to be in the space of bounded deformation histories.

Chacon and Rivlin [27], by making use of the Stone-Weierstrass approximation theorem [28], have shown that any continuous functional can be uniformly approximated on a subset of D by a polynomial in linear functionals from L, where D is a topological real Hausdorff vector space of tensor-valued functions and L is a subspace of functionals which distinguishes elements in D.

Lew [29] improved upon the results of Chacon and Rivlin through a more subtle use of the Stone-Weierstrass theorem. Here the conditions imposed upon the functional are less restrictive than those imposed by Chacon and Rivlin. In particular, Lew shows that the functional must only be uniformly continuous in the weak topology defined on D by L.

T. T. Wang [30] derived the integral representation by means of Gateaux's theory of functional representations [31]. Gateaux's method seems to provide an adequate description of the phenomenological processes taking place within the material while at the same time yielding the standard form of the integral approximation.

In the following section we will present a particularly simple formal method for deriving the integral approximation.

D. The Black Box Problem

The black box (Fig. 10) is defined as anything which acts upon an input and produces an output [32]. It is not known why boxes are black, but as Wiener [33] states, "boxes are ex officio black." The black box problem was first formulated, in its general form, by electrical engineers. An electrical engineer is given a sealed black box containing an unknown assembly of electronics. The black box has terminals for applying inputs and other terminals for outputs. The engineer may then apply any type of electrical input he can generate and then measure any output his equipment is capable of measuring. His problem is to determine the contents of the black box by this method.

Problems of this nature are of fundamental importance in many scientific fields. In fact, much of the research on the black box problem is being carried out in the life sciences.

The analog of the black box problem in the mechanics of continua can be posed as follows. Suppose we are given a black piece of some unknown material, then the problem is to determine the nature of the material by applying certain inputs (stresses, strains) and by measuring the corresponding outputs (strains, stresses). To be more specific, we must determine the constitutive equation of the unknown material by experiment.

Let us take an unknown, possibly nonlinear, material which is represented by the black box in Fig. 10. Assume that the input to the material is the rate of change of strain with respect to the arc length and that the corresponding output is the stress. This means that the constitutive relation (3.11) describes this process. In general, the present state of stress of a nonlinear material depends upon the strain rate history in a nonlinear manner.

The history of the strain rate can be expressed in terms of a system of quantities, which may be infinite in number, and which exist at the present time. No assumption is made concerning the nature of these quantities. All that is being said is that the history of the input to the material is expressible in terms of certain quantities. Since the material is nonlinear, the present output, or the state of stress, may be expressed in terms of a nonlinear operation upon this set of quantities.

Let us see what can be said about the relation between input and output without any further assumptions. This problem has been studied by many authors [33], [34], [35], [36].

We have asserted that the strain rate history $\dot{E}(s)$ can be described by means of the quantities,

$$W_0(\varsigma), W_1(\varsigma), W_2(\varsigma), \ldots, \tag{3.15}$$

evaluated at "time" $\S$. Thus the present state of stress may be written as

$$\pi(\S) = G[W_0(\S), W_1(\S), W_2(\S), \ldots] \tag{3.16}$$

where G denotes some nonlinear operation.

Let $\{h_i(s)\}$ be any complete orthonormal set of functions, i.e.

$$\int_0^{\S} h_i(s) h_j(s)\, ds = \begin{Bmatrix} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{Bmatrix} . \tag{3.17}$$

$\dot{E}(s)$ at any time in the past may now be expressed as an expansion in these orthonormal functions,

$$\dot{E}(s) = \sum_{i=0}^{\infty} W_i(\S) h_i(s) \tag{3.18}$$

where $W_i(\S)$ is the coefficient of the i-th orthonormal function in the expansion. Since the set $\{h_i(s)\}$ is composed of known functions, the coefficients $\{W_i(\S)\}$ completely describe $\dot{E}(s)$ from the remotest past up to the present time. Thus, if the coefficients are known we can reconstruct the entire strain history.

The unknowns, in this case being the coefficients $\{W_i(\S)\}$, may be determined as follows. Multiply (3.18) by $h_j(s)$ and obtain,

$$h_j(s)\dot{E}(s) = \sum_{i=0}^{\infty} W_i(\S) h_i(s) h_j(s) . \tag{3.19}$$

Applying the orthogonality property (3.17) and integrating (3.19) over s yields,

$$W_j(\S) = \int_0^{\S} h_j(s) \dot{E}(s)\, ds . \tag{3.20}$$

In general $\{W_i(\S)\}$ will be an infinite set.

Now assume that the nonlinear operation G in (3.16) is a general polynomial in the $W_i(\S)$. Thus the Piola-Kirchoff stress tensor may be written as,

$$\Pi(\S) = a + \sum_{i=0}^{\infty} b_i W_i(\S) + \sum_{i,j=0}^{\infty} c_{ij} W_i(\S) W_j(\S) + \ldots\ldots \\ + \sum_{i,j,\ldots=0}^{\infty} g_{ij\ldots\ldots} W_i(\S) W_j(\S) \ldots\ldots + \ldots\ldots \tag{3.21}$$

Here the coefficients a, b_i, c_{ij}, ..., g_{ij}, ..., ..., completely describe the nonlinear operation, and we have the stress state in terms of the strain rate history. Now substitute (3.20) into (3.21) and the polynomial becomes,

$$\Pi(\S) = a + \sum_{i=0}^{\infty} b_i \int_0^{\S} h_i(s_1) \dot{E}(s_1)\, ds_1 + \\ + \sum_{i,j=0}^{\infty} c_{ij} \int_0^{\S} h_i(s_1) \dot{E}(s_1)\, ds_1 \int_0^{\S} h_j(s_2) \dot{E}(s_2)\, ds_2 + \ldots\ldots \tag{3.22}$$

The order of integration and summation is now interchanged to obtain,

$$\Pi(\S) = a + \int_0^{\S} ds_1 \dot{E}(s_1) \sum_{i=0}^{\infty} b_i h_i(s_1) + \\ + \int_0^{\S} ds_1 \int_0^{\S} ds_2 \dot{E}(s_1) \dot{E}(s_2) \sum_{i,j=0}^{\infty} c_{ij} h_i(s_1) h_j(s_2) + \ldots\ldots \tag{3.23}$$

We may, however, write the summations as,

$$\sum_{i,j\ldots=0}^{\infty} g_{ij\ldots} h_i(s_1) h_j(s_2) \ldots = K(s_1, s_2, \ldots). \tag{3.24}$$

Substituting (3.24) into (3.23) yields,

$$\Pi(\mathcal{S})=a+\sum_{n=1}^{\infty}\int_{0(n)}^{\mathcal{S}} K_n(s_1,s_2,\ldots,s_n)\dot{E}(s_1)\ldots\dot{E}(s_n)ds_1\ldots ds_n. \qquad (3.25)$$

The nonlinear system is now completely described by the kernels $K_n(s_1,\ldots,s_n)$. If it assumed that the stress state of the material is zero at $t=0$, then the constant term, a , must vanish in equation (3.25) and equation (3.26) is obtained

$$\Pi(\mathcal{S})=\sum_{n=1}^{\infty}\int_{0(n)}^{\mathcal{S}} K_n(s_1,s_2,\ldots,s_n)\dot{E}(s_1)\ldots\dot{E}(s_n)ds_1\ldots ds_n . \qquad (3.26)$$

E. The Relation to Plastic Work Hardening

The arc length parameter we have been using may be compared to the strain hardening parameter of plasticity theory. In the classical theory of plasticity the total strain E_{ij} is decomposed into a plastic strain P_{ij} and an elastic strain Q_{ij} . The elastic strains are related to the stresses by Hooke's law. The relation between the plastic strains P_{ij} and the stresses are given by the constitutive relation

$$\dot{P}_{ij}=\begin{cases} 0 & \text{for unloading} \\ f(\Pi'_{ij},P_{ij})\dot{\Pi}'_{ij} & \text{for loading} \end{cases} \qquad (3.27)$$

where Π_{ij} denotes the deviatoric stress tensor.

Thus it can be seen that in order to make use of a constitutive equation of the type (3.27) it becomess necessary to give precise meanings to the terms loading and unloading, especially for general three dimensional programs. In our theory, as can be seen from (3.26) there is no need to arbitrarily define a process as a loading or an unloading process. This is because the terms in (3.26) which involve the strain rate have components which change sign when the direction of straining is reversed. Thus the relation (3.26) automatically takes care of specifying whether or not a particular process is a loading or an unloading process. Also each component of the strain rate matrix may change sign independently of the other components. Indeed, processes in which the signs of the components of the strain rate matrix change arbitrarily and at different times are possible within the formulation (3.26) without having to define a particular process as an unloading or a loading process.

The plastic strain hardening parameter z is defined in terms of the plastic strain rate,

$$z = \int_0^{\tau} (\dot{P}_{ij}\, \dot{P}_{ij})^{1/2}\, d\tau' \quad . \tag{3.28}$$

The yield condition is assumed to be of the form,

$$f(\pi', z) = F(\pi') - H(z), \tag{3.29}$$

where the temperature is not taken into consideration.

A deformation process is said to be strain hardening if,

$$\frac{\partial H}{\partial z} \geq 0 \quad . \tag{3.30}$$

A deformation is defined to be a loading process when,

$$f = 0, \ \dot{f} = 0, \ \text{and} \ \dot{z} > 0 \quad . \tag{3.31}$$

The loading condition may also be expressed as

$$\frac{\partial f}{\partial \pi_{ij}} \dot{\pi}_{ij} > 0 \tag{3.32}$$

If the onset of plastic flow is described by means of the von Mises yield condition,

$$f(\pi'_{ij}) = \pi'_{ij}\pi'_{ij} - k^2 = 0 \tag{3.33}$$

then the plastic strain rate may be determined by means of the following relation

$$\dot{P}_{ij} = \frac{\pi'_{ij}}{|\pi'_{ij}| \frac{\partial H}{\partial z}} \cdot \frac{\partial f}{\partial \pi'_{kn}} \dot{\pi}'_{kn} \tag{3.34}$$

where it is assumed that the material is incompressible and independent of temperature.

The above discussion of strain hardening plasticity points out the similarity between the arc length parameter S and the strain hardening parameter Z. It also points out the advantage of our formulation (3.26) in that we have no need to define a process as being either loading or unloading.

In (3.26) consider only the linear term of the expansion. In this case the kernel function can be suitably chosen in order to achieve equations similar to those of strain hardening plasticity. In fact, if we work with only the plastic strains and deviatoric tensors a theory equivalent to strain hardening plasticity can be obtained. If the kernel function is chosen to be a Dirac delta function a special form of strain hardening plasticity is obtained.

The object of both the arc length and strain hardening parameters is the introduction of some degree of irreversibility into the description of the mechanical behavior of the material. A consequence of this irreversibility is that a stress-strain curve $\pi = f(E)$ cannot be retraced even if the variation of the strain E is reversed.

We have here, in effect, asserted that a given rate independent material will have its own intrinsic time, S or Z, associated with it. And by means of this time, which will in general be different from the actual time, we can more naturally represent the mechanical behavior of the material. That is, the constitutive equation expressed in terms of the intrinsic time will be a more natural representation of the material than a constitutive equation expressed in terms of the actual time.

IV. THE SCALAR THEORY

A. The Straining Program

In this section a one dimensional theory is developed whose purpose it is to obtain a relation between the scalar stress and the scalar strain. Thus, this theory is an attempt to describe the stress-strain relation which is obtained in the laboratory. This stress-strain relation is obtained by performing a simple uniaxial tension or compression test. The result of just such a cyclic test is illustrated in Fig. 2.

In order to completely describe a one dimensional cyclic stress-strain relation, two strain parameters must be prescribed. The first is the maximum strain, A, achieved prior to unloading, and the second is the permanent set $E^*(A)$. If more than one unloading will be considered, then these two parameters must be prescribed for each loading cycle.

Assume that a rate independent material has been strained at some given rate r, and that this straining process has taken the material from its undeformed state O to some value of strain, A, (see Fig. 2). The effect of this deformation process is best seen by observing the stress-strain curve for the material which begins at the origin O, and ends at point A.

Instead of straining the material at rate r, assume that the material had been strained at the rate r'. If this new deformation process also takes the material from its undeformed state O, to the point A, then the same locus of stress-strain points, obtained above by deforming the material at rate r, would have been obtained. Indeed, for a rate independent material, any straining program which deforms the material from its undeformed state O, to the point A, monotonically, will trace out the same unique path. Similarly, for the release of strain, any rate p of release of strain will be equivalent to any other rate p' of release of strain. This means the path between A and $E^*(A)$ in Fig. 2, is independent of the rate of release of strain. Also, the restraining path will be rate independent and so on for successive processes. Therefore, without any loss in generality, the straining program, illustrated in Fig. IIa, may be hypothesized for the cyclic stress-strain test.

The straining program shown in Fig. IIa is defined by,

$$E(\tau) = m\tau \qquad 0 \leq \tau \leq \tfrac{1}{2} \tag{4.1}$$

$$E(\tau) = m\left[1 - \frac{k}{2} + (k-1)\tau\right], \qquad \tfrac{1}{2} \leq \tau \leq 1 \tag{4.2}$$

$$E(\tau) = m\left[\tau - 1 + \frac{k}{2}\right] \qquad \tau \geq 1 \tag{4.3}$$

where for our purposes it is assumed that

$$0 < k \leq 1 \text{ and } m > 0 \ . \tag{4.4}$$

For this program $\dot{E}$ changes sign at $\tau = \frac{1}{2}$ and again at $\tau = 1$. Built into this program is the condition that the material will not return to its undeformed state, when the applied load is removed. The extension of the material, relative to its undeformed state, observed when the applied load has been removed is called the permanent set, or the residual deformation. This corresponds to the actual behavior of graphite which has been previously described.

In Fig. IIa the relative maximum strain occurs at $\tau = \frac{1}{2}$ and its value is $m/2$. The permanent set E^* is the value of the strain at $\tau = 1$, or

$$E^* = k(m/2). \tag{4.5}$$

Thus k represents the fraction of the relative maximum strain which remains as the permanent set.

It can be seen from (4.2) that if k were allowed to be unity we would have

$$E = m/2 \tag{4.6}$$

in the time interval $\frac{1}{2} \leq \tau \leq 1$. This means that, independently of time, the value of the relative maximum strain would be retained. In short, when $k=1$ the material experiences no elastic recovery.

If k were to vanish, then we would have from (4.2),

$$E = m(1-\tau) \tag{4.7}$$

This means that there would be no permanent set for $\tau=1$ and hence the deformation could be considered as completely recoverable.

Let us look at some of the properties of the program (4.1) to (4.4). By equation (3.7) the value of the arc length for straining is,

$$S(\tau) = \int_0^{\tau} \sqrt{\frac{d(m\tau')}{d\tau'} \cdot \frac{d(m\tau')}{d\tau'}}\; d\tau' \tag{4.8}$$

$$= \int_0^{\tau} m\, d\tau' = m\tau \quad . \tag{4.9}$$

Thus for straining, $\tau = s/m$ when

$$0 \leq s \leq m/2 \qquad \text{and} \qquad 0 \leq \tau \leq 1/2 \tag{4.10}$$

The value of the arc length parameter for the release of strain process is,

$$s(\tau) = m\tau\Big|_{\tau=\frac{1}{2}} + \int_{1/2}^{\tau} \left[\frac{d}{d\tau'} \left\{ m\left[1-\frac{k}{2}+\tau'(k-1)\right] \right\} \cdot \right. \tag{4.11}$$

$$\left. \frac{d}{d\tau'} \left\{ m\left[1-\frac{k}{2}+\tau'(k-1)\right] \right\} \right]^{1/2} d\tau' = \frac{m}{2} + \int_{1/2}^{\tau} \sqrt{m^2(k-1)^2}\ d\tau' =$$

$$= \frac{m}{2} + m(1-k)\tau\Big|_{1/2}^{\tau} = \frac{m}{2} + m(1-k)\tau - \frac{m}{2}(1-k) =$$

$$= \frac{mk}{2} + m(1-k)\tau \tag{4.12}$$

Note that the sign of the square root of $m^2(k-1)^2$ in (4.9) was chosen so that the arc length would always be positive and hence would always be a monotonic increasing function of time. For the arc length to be otherwise would be meaningless. Thus for the release of strain process the arc length is given by (4.12) when

$$m/2 \leq s \leq (m-km/2) \quad \text{and} \qquad 1/2 \leq \tau \leq 1 \tag{4.13}$$

where k is less than one and positive. Rewriting (4.12) we have

$$\tau = \frac{2s - km}{2m(1-k)} \quad , \tag{4.14}$$

in the time interval $\frac{1}{2} \leq \tau \leq 1$.

For the re-straining process we have

$$s(\tau) = \left[\frac{mk}{2} + m\left(1 - \frac{k}{2}\right)\tau\right|_{\tau=1} + $$

$$+ \int_1^{\tau} \sqrt{\frac{d}{d\tau'}\left[m\left(\tau - 1 + \frac{k}{2}\right)\right]\frac{d}{d\tau'}\left[m\left(\tau - 1 + \frac{k}{2}\right)\right]}\, d\tau' \tag{4.15}$$

or

$$s(\tau) = m\tau - m\left(\frac{k}{2}\right) \tag{4.16}$$

and

$$\tau = \frac{s}{m} + \frac{k}{2}, \quad \text{for } \tau > 1 \text{ and } s > m - m\left(\frac{k}{2}\right).$$

At this point $E(s)$ can be written for the cases of straining, release of strain, and re-straining.

For straining, $E = m\tau = s, \quad 0 \leq s \leq \frac{m}{2}$ (4.17)

For strain release, $E = m - s, \ \frac{m}{2} \leq s \leq m\left(1 - \frac{k}{2}\right)$ (4.18)

For re-straining, $E = m\left(\frac{s}{m} + k - 1\right), \quad s \geq m\left(1 - \frac{k}{2}\right)$ (4.19)

Thus given a monotonic increasing straining process and a monotonic decreasing release of strain process, only the relative maximum strain parameter, $m/2$, and the value of the permanent set,

$^{mk}/_{2}$, need to be given in order to completely describe the history of the deformation process. This means that when we specify m and k we specify the strain history.

B. Linear Term Analysis

Let us look at the previously derived integral approximation (3.26). We assume that the material under consideration is stress free in its undeformed state. From (4.17), (4.18), and (4.19) we see that $\dot{E}(s)=1$ for straining and $\dot{E}(s)=-1$ for the release of strain. In order for (3.26) to accurately describe a program of the type shown in Fig. IIa, it should feel the effects of the release of strain. The even ordered terms in (3.26) will not, however, feel these effects, i.e. the even ordered terms in (3.26) cannot tell the difference between a straining and a release of strain process. Thus we are led to conclude that if we wish to accurately model the mechanical behavior of graphite by means of (3.26), the even ordered terms in (3.26) should be omitted.

There exists a strong possibility that certain types of graphite may need, aside from the linear term, higher terms in order to accurately describe the behavior of graphite under cyclic straining programs. Here, however, we week to ascertain as to whether or not the linear term in (3.26) is capable of describing the qualitative mechanical properties exhibited by graphite under cyclic straining programs.

The linear term in the approximation (3.26) is,

$$\pi(\mathcal{S})=\int_{0}^{\mathcal{S}} k(s)\,\dot{E}(s)\,ds \tag{4.20}$$

The strain history is known from experiment and therefore $\dot{E}(S)$ may be found by applying (4.17), (4.18), or (4.19). Now the form of the kernel function must be determined. Equation (4.20) states that an increment in strain, $dE(S)$ produces an increment in stress

$$d\pi(\S) = k(S)dE(S) \tag{4.21}$$

which is independent of the values of the strain outside of the interval $(S, S+dS)$.

From our experience with graphite we can make a good guess as to the form of the kernel function. If $k(S)$ is constant, then we have,

$$\pi(\S) = k'E(\S) \tag{4.22}$$

where k' is a constant. This is a theory of elasticity, since the stress is a function of the strain.

Now if $k(S)$ is a Dirac delta function, then we obtain,

$$\pi(\S) = k''\dot{E}(\S), \tag{4.23}$$

where k'' is a constant. This is a form of work hardening plasticity, where the stress is a function of the strain rate. Both (4.22) and (4.23) are inadequate for our purpose.

We now assume the form of the kernel function $k(S)$ which will be used in the subsequent analysis. In the discussion that immediately follows, one must always keep in mind that while the variable, S, is time-like due to the monotonic property of S, it is a function of the applied strains and not of time.

The form of the kernel we will employ to model the cyclic one dimensional mechanical response of graphite is,

$$K(s) = c \cdot \exp\{[a + bn(1-k)]\,\hat{s} - bs\} \tag{4.24}$$

where n is the number of complete reversals in the straining process and $\hat{s}$ is the value of the arc length parameter which corresponds to the previous maximum strain experienced by the graphite specimen. The motivation for choosing the kernel function (4.24) is due to the consideration of two facts. First, Woolley [8] showed that a one dimensional constitutive equation in exponential form was able to describe the uniaxial loading response of graphite exceedingly well. Second, the kernel function must be chosen so that the dependence of graphite on the previously attained maximum strain is incorporated into the constitutive equation. This is accomplished by including n and $\hat{s}$ in the constitutive equation.

At this point let us see how well or poorly some other kernel functions, similar to (4.24), model the mechanical behavior of graphite. Also, by analyzing the properties of these kernel functions we shall see where they fail in describing the mechanical properties of graphite and obtain an indication as to what is needed in order to accurately describe graphite's mechanical behavior. It is partly by analyses of this type that the kernel function (4.24) was chosen.

Let us consider a simplified form of the kernel function (4.24),

$$K(s) = c \cdot \exp(-bs). \tag{4.25}$$

With this choice of kernel function the linear constitutive relation (4.20) for straining, becomes

$$\pi(\xi) = c\int_0^{\xi} \exp(-bs)\,ds\,, \tag{4.26}$$

where we have used $\dot{E}(s) = +1$ for straining. By carrying out the operation indicated by (4.26) we obtain

$$\pi(\xi) = \frac{c}{b}\left[1 - \exp(-b\xi)\right]\,. \tag{4.27}$$

It is at once evident that (4.27) is essentially the same result as (2.8) obtained by Woolley from dislocation theory considerations. Woolley has shown that a stress-strain relation having the form of (4.27) is an excellent fit to the straining compressive stress-strain data for graphite.

Let us see how well a representation of the type (4.27) will represent the mechanical behavior of graphite for the release of strain process. For the release of strain we have

$$\pi(\xi) = \frac{c}{b}\left[1 - \exp(-\tfrac{1}{2}bm)\right] + c\int_{m/2}^{\xi} \exp(-bs)(-1)\,ds\,. \tag{4.28}$$

In (4.28) the term $\dot{E}(s) = -1$ corresponds to the release of strain process and $s = m/2$ when $\tau = 1/2$. Integrating (4.28) and combining terms yields,

$$\pi(\xi) = \frac{c}{b}\left[1 - 2\exp(-\tfrac{1}{2}bm) + \exp(-b\xi)\right]\,. \tag{4.29}$$

While, as we have noted above, (4.27) will accurately describe the stress-strain curve for graphite for the initial straining, the counterpart of (4.27) for the release of strain process (4.29) is not an accurate description of the observed phenomena. The reason for this is shown in Fig. 12. In Fig. 12 we see that the strain release curve intersects the straining curve well above the stress axis. This is a phenomenon which is never observed in graphite. Thus we conclude that the kernel function (4.25) fails in describing the cyclic straining behavior of graphite.

The next kernel function we consider will also be seen to fail at describing the mechanical response of graphite. It is presented here, however, because it is felt that a representation using this kernel function can be applied to many other materials.

This kernel function is,

$$K(\S - s) = c \cdot \exp(a\S - bs). \tag{4.30}$$

The linear functional (4.20) with the kernel function (4.30) can be thought of as a linear theory of rate independent viscoelasticity. Whereas a viscoelastic material will have a fading memory of deformation, our material has a fading memory of the rate of change of strain with respect to the arc length.

With this kernel function we have for the initial straining,

$$\pi(\S) = c\int_0^{\S} \exp(a\S - bs)(+1)\, ds \tag{4.31}$$

$$= \frac{c}{b} \exp(a\S)\left[1 - \exp(-b\S)\right]. \tag{4.32}$$

In order for the stress-strain relation (4.32) to have the properties exhibited by room temperature reactor grade graphite it is necessary that the second derivative of the stress be less than zero for all values of $\S > 0$.

From the condition that the first derivative be greater than zero we obtain,

$$\exp(-b\S) < a/(a-b) \quad . \tag{4.33}$$

This implies that

$$b > 0 \tag{4.34}$$

For the second derivative we have

$$\frac{d^2\Pi}{dE^2} = \frac{d^2\Pi}{ds^2} = \frac{c}{b}\left[a^2\exp(a\S) - (a-b)^2\exp[(a-b)\S]\right] < 0 \tag{4.35}$$

and this inequality may be reduced to

$$\frac{a^2}{(a-b)^2} < \exp(-b\S) \quad . \tag{4.36}$$

The left hand side of the inequality (4.36) is always greater than zero. Since b, is positive and $\S$ is monotonic increasing the right hand side of (4.36) tends to zero exponentially in $\S$. Thus no matter how small the constant on the left hand side of (4.36) is, the right hand side of the inequality will eventually, with increasing arc length, become small enough to violate the inequality (4.36). This implies that at some point the stress-strain curve described by (4.32) will become concave. Actually the stress-strain curves for many materials, especially ductile metals, exhibit this type of phenomena. Graphite on the other hand, being a brittle material does not exhibit phenomena of this type. It is observed that the stress-strain curve for graphite remains convex until fracture. It is felt that the kernel function (4.30) might provide a suitable representation for rate independent ductile materials.

At this point in our investigation let us return to the kernel function (4.24) which we have chosen to represent the mechanical response of graphite. For straining $\dot{E}(s) = +1$, n and $\hat{\S}$ will be zero. Thus we may write

$$\Pi(\S) = c\int_0^{\S} \exp(-bs)(+1)\,ds =$$

$$= \frac{c}{b}\left[1 - \exp(-b\S)\right]. \tag{4.37}$$

For the straining process we have, $E(\S) = \S$, and the first derivative of the stress may be written as,

$$\frac{d\Pi}{dE} = \frac{d\Pi}{d\S} = c\ \exp(-b\S). \tag{4.38}$$

For graphite the first derivative (4.38) must always be greater than zero for all $\S$. Thus for $\S = 0$ we have,

$$\left.\frac{d\Pi}{d\S}\right|_{\S=0} = c > 0 \tag{4.39}$$

and the constant c is interpreted as the tangent modulus of the graphite specimen at $\mathcal{S} = 0$, i.e., at the origin of the stress-strain curve.

From the condition that the second derivative of the stress with respect to the strain,

$$\frac{d^2\pi}{dE^2} = \frac{d^2\pi}{d\mathcal{S}^2} = -bc \exp(-b\mathcal{S}) \tag{4.40}$$

should always be negative we obtain the condition that $b > 0$. The value of the constant b can now be chosen so that the best fit to the monotonic increasing straining portion of the graphite stress-strain curve is obtained.

For the release of strain process, $\dot{E}(S) = -1$, $\hat{\mathcal{S}} = \frac{m}{2}$, and $n = 1$ for this first strain reversal. Here $\hat{\mathcal{S}} = \frac{m}{2}$ is the arc length (strain) corresponding to the point strain reversal. Thus,

$$\begin{aligned} \pi(\mathcal{S}) &= \int_0^{m/2} c \exp(-bs)(+1)\,ds + \int_{m/2}^{\mathcal{S}} c \exp\{[a+nb(1-k)]\hat{\mathcal{S}} - bs\}(-1)\,ds \\ &= \pi(\tfrac{m}{2}) + \tfrac{c}{b} \exp\{[a+b(1-k)]\tfrac{m}{2}\}\left[\exp(-b\mathcal{S}) - \exp(-\tfrac{bm}{2})\right]. \end{aligned} \tag{4.41}$$

Now, since for the release of strain

$$\frac{d\pi}{dE} = -\frac{d\pi}{d\mathcal{S}} = c \exp\{[a+b(1-k)]\tfrac{m}{2}\} \exp(-b\mathcal{S}) > 0 \tag{4.42}$$

the constant c must be greater than zero, and the condition that the second derivative

$$\frac{d^2\pi}{dE^2} = \frac{d^2\pi}{d\hat{s}^2} = bc\ \exp\left\{\left[a+b(1-k)\right]\frac{m}{2}\right\}\exp(-b\hat{s}) > 0 \qquad (4.43)$$

be greater than zero yields the condition that b must be greater than zero. These are the same conditions which were obtained for the straining process. There is, however, the additional condition that the stress return to zero at the permanent set. That is, when the strain is $k(\frac{m}{2})$ (or the arc length reaches $m-k(\frac{m}{2})$) the stress must be zero. Thus,

$$0 = \pi(\tfrac{m}{2}) + \frac{c}{b}\exp\left\{\left[a+b(1-k)\right]\frac{m}{2}\right\}\left\{\exp\left[-b(m-\tfrac{km}{2})\right] - \exp(\tfrac{-bm}{2})\right\}$$

or

$$\pi(\tfrac{m}{2}) = \frac{c}{b}\exp(\tfrac{2m}{2})\left[\exp(\tfrac{-bkm}{2}) - \exp(\tfrac{-bm}{2})\right] . \qquad (4.44)$$

For given values of $\pi(m/2)$, c , b, k, and m , equation (4.44) will determine the third material parameter, a . By using (4.37) we can obtain the value of $\pi(m/2)$, substituting this value into (4.44) yields the following relation for determining a ,

$$\exp(\tfrac{2m}{2}) = \frac{1-\exp(\tfrac{-bm}{2})}{\exp(\tfrac{-bkm}{2}) - \exp(\tfrac{-bm}{2})} . \qquad (4.45)$$

For the process of re-straining $\dot{E}(s)=+1$, $n=2$, and $\hat{s}$ retains the value of $m/2$. Hence,

$$\pi(\hat{s}) = \pi\left(m-\frac{km}{2}\right)+\int_{m(1-\frac{k}{2})}^{\hat{s}} c \exp\left\{\left[a+2b(1-k)\right]\frac{m}{2} - bs\right\}(+1)\,ds$$

$$= 0+\frac{c}{b}\left\{\exp\left[\frac{m}{2}(a-kb)\right]-\exp\left[\frac{am}{2}+\frac{b}{m} - kbm - b\hat{s}\right]\right. \quad . \tag{4.46}$$

When the graphite specimen has been re-restrained to the point of the previous maximum strain the corresponding arc length is

$$s = m\left(3/2 - k\right) \quad . \tag{4.47}$$

Thus

$$\pi\left(\frac{3m}{2}-km\right)=\frac{c}{b}\exp\left(\frac{am}{2}\right)\left[\exp\left(-\frac{kbm}{2}\right)-\exp\left(\frac{-bm}{2}\right)\right]=\pi\left(\frac{m}{2}\right) \tag{4.48}$$

where this result follows from (4.44).

C. Comparison to Experiment

We begin by assigning values to the constants found in the kernel function (4.24). The constant c has been shown to be the initial tangent modulus. For the sake of comparison let the constant c = 428 lbs. Since graphite is a rate independent material any one rate of straining is equivalent to any other rate of straining. Therefore let m = 1 in some suitable unit, say 1/sec. Let b = 1/2 be the best fit to the monotonic straining portion of the graphite stress-strain curve. The permanent set is assumed to be 0.1 in our units of strain. Assume that our strain measure multiplied by 3570 yields the strain in units of μ in./in. With the above values for the constants we find that a is approximately 0.53. The constant $\hat{s}$ is equal to 1/2 for both the release of strain and re-straining processes and is zero for straining. The value of n is zero for straining, one for the release of strain, and two for restraining. Thus we have, for straining,

$$c = 428 \text{ lbs.}, \quad m = 1, \quad b = 1/2, \quad k = 0.2, \quad \hat{\$} = n = 0.$$

For the release of strain and re-straining the first four equalities above are retained, but we have for the strain release process,

$$n = 1, \quad \hat{\$} = 1/2,$$

and for the restraining process,

$$n = 2, \quad \hat{\$} = \tfrac{1}{2} .$$

Now for straining we may write,

$$k(s) = c \cdot \exp(-bs) \tag{4.49}$$

or

$$\pi(\$) = \frac{c}{b}\left[1 - \exp(-b\$)\right] = 856\left[1 - \exp(\$/2)\right]. \tag{4.50}$$

For straining $E = \$$, thus we may write down the values in table I.

TABLE I

STRAIN μin/in	$E = \$$	$e^{-\$/2}$	$1-e^{-\$/2}$	$\pi(\$)$ lbs.
000.0	0.0	1.000	0.0000	0.0000
357.0	0.1	0.951	0.0488	41.73
714.0	0.2	0.905	0.0952	81.45
1071.0	0.3	0.861	0.1393	119.24
1428.0	0.4	0.819	0.1813	155.15
1785.0	0.5	0.779	0.221	189.35

The release of strain process is associated with the following kernel function,

$$K(s) = 428 \exp\{[0.53 + \tfrac{1}{2}(0.8)]^{\frac{1}{2}} - bs\} \tag{4.51}$$

and the stress for this strain release process is given by,

$$\pi(\mathcal{S}) = 189.35 + 1361.04 \int_{1/2}^{\mathcal{S}} \exp(-\tfrac{1}{2}\mathcal{S})(-1)\, ds \tag{4.52}$$

$$\pi(\mathcal{S}) = 189.35 + 1361.04 \left[\exp(-\tfrac{1}{2}\mathcal{S}) - \exp(-\tfrac{1}{4})\right] \tag{4.53}$$

$$\pi(\mathcal{S}) = -870.87 + 1361.04 \exp(-\tfrac{1}{2}\mathcal{S}) . \tag{4.54}$$

For the strain release process the arc length is equal to neither the time variable or the strain, but is from (4.12),

$$\mathcal{S} = \tfrac{1}{2} mk - m(1-k)\tau \tag{4.55}$$

or

$$\mathcal{S} = 0.1 + 0.8\tau . \tag{4.56}$$

The strain is given by (4.18) and is,

$$E = 1 - \mathcal{S} . \tag{4.57}$$

Thus we may now write the values of the stress corresponding to the strain inputs as shown in table II.

TABLE II

STRAIN μin/in	τ	ξ	E	$e^{-\xi/2}$	$\pi(\xi)$ lbs.
1,499.4	0.6	0.58	0.42	0.7483	141.24
1,213.8	0.7	0.66	0.34	0.7189	101.01
928.2	0.8	0.74	0.26	0.6907	62.70
642.6	0.9	0.82	0.18	0.6636	25.69
357.0	1.0	0.90	0.10	0.6376	9.4

For re-straining the kernel function becomes,

$$K(s) = 428 \exp\{[0.53 + (0.8)]\tfrac{1}{2} - bs\} \qquad (4.58)$$

it then follows that the stress is given by

$$\pi(\xi) = 428\int_{0.9}^{\xi} \exp(0.665 - bs)\, ds \qquad (4.59)$$

$$\pi(\xi) = 856\ (1.944)\left[\exp(-0.45) - \exp(-\tfrac{1}{2}\xi)\right] \qquad (4.60)$$

$$\pi(\xi) = 356\left[1.24 - 1.94\exp(-\tfrac{1}{2}\xi)\right]. \qquad (4.61)$$

For re-straining, however, by (4.16)

$$\xi = m\tau - \tfrac{1}{2}mk \qquad (4.62)$$

or

$$\xi = \tau = 0.1 \qquad (4.63)$$

and by (4.19)

$$E = \xi - 0.8 \qquad (4.64)$$

Thus we have the values shown in table III.

TABLE III

STRAIN μ in/in	τ	§	E	$e^{-§/2}$	$\pi(§)$ lbs.
714.0	1.1	1.0	0.2	0.6065	55.36
1071.0	1.2	1.1	0.3	0.5769	103.32
1428.0	1.3	1.2	0.4	0.5488	149.80
1785.0	1.4	1.3	0.5	0.5220	194.30

The stress-strain curve plotted from these values is shown, in Fig. 13, to closely reproduce an actual experimental graphite stress-strain curve for cyclic straining. Thus our model appears to be a suitable representation for the cyclic straining behavior of reactor grade polycrystalline graphite.

V. THE THREE DIMENSIONAL THEORY

A. Invariants of the Transverse Isotropy Group

In this section the basic theory of the invariants of the transverse isotropy group is given [37], [38], [39]. We start with the definition of a tensor invariant. Let A be an arbitrary tensor with n components $A_1, A_2, \ldots, A_n$. If these components are mapped onto the components $A^*_1, A^*_2 \ldots A^*_n$ by some linear transformation of space, then a function $H(A_1, \ldots, A_n)$ of the tensor components with the property that

$$H(A^*_1, \ldots, A^*_n) = \Delta^g \; H(A_1, \ldots, A_n) \qquad (5.1)$$

is called an invariant of the tensor A. In (5.1) Δ is called the determinant of the transformation and g is called the weight of the determinant. When $g=0$ the invariant H is termed an absolute invariant and when $g \neq 0$ the invariant is called a relative invariant.

Analogously, if $\{G\}$ is an arbitrary group of linear transformations L, in an n-dimensional vector space V, then H is an absolute invariant of $\{G\}$, if for every linear transformation L in $\{G\}$,

$$H(LA_1, \ldots, LA_n) = H(A^*_1, \ldots, A^*_n) = H(A_1, \ldots, A_n). \qquad (5.2)$$

The set of functions $H_i(A)$ which are invariants of $\{G\}$ for each i, form an integrity basis for $\{G\}$.

If we are given an arbitrary function H' on V which is also invariant under $\{G\}$, then H' may be expressed as some function of the functions $H_i(A)$. Thus for a function of a single variable,

$$H'(A) = J(H_1(A), H_2(A), \ldots) \tag{5.3}$$

and for a function of many variables,

$$H'(A, B, C, \ldots) = J(H_1(A, B, C, \ldots), \ldots) \tag{5.4}$$

where J is a function of the indicated variables.

It is worthwhile to note that the integrity basis, as defined above, is a function basis which is, in general, different from the usual basis for a n-dimensional vector space.

We can now state the most important theorem in the theory of invariants which is due to D. Hilbert [40]. A quantic in any number of variables has a finite system of independent invariants.

The transverse isotropy group is defined as that continuous group of motions such that all directions in a material which are perpendicular to the axial direction h are equivalent. It is obvious that the transverse isotropy group is a subgroup of the orthogonal group. There are various types of transverse isotropy depending on whether or not certain reflections are permitted as symmetry operations [41]. Graphite exhibits the type of transverse isotropy which is characterized by the admission of reflections in the planes perpendicular to the h-axis as symmetry operations. This implies that vectors of the type (0, 0, h) will be mapped onto vectors of the tyoe (0, 0, -h) by the reflections.

We now state two basic theorems concerning the determination of the invariants of the transverse isotropy group. First, if the function H on V is an invariant of $\{G\}$, then H may be expressed as a function of a finite number of invariants of $\{G\}$. This is a consequence of Hilbert's theorem. Second, if $\{G\}$ is the full orthogonal group, then the complete table of invariants can be expressed in terms of

$$(u,v) \text{ and } [u,v,\ldots,w] \tag{5.5}$$

where $u,v,\ldots,w$ are vectors in V, (u,v) denotes the scalar product of u and v, and $[u,v,\ldots,w]$ denotes the determinant of the vectors $u,v,\ldots,w$.

B. Material Symmetry

In order to achieve a completely general description of the mechanical properties of graphite, the constitutive equation for graphite must be written in a form which will exhibit the symmetry properties of the material. In our case the problem reduces to finding the invariants of a system of second order tensors under the transversely isotropic group of transformations.

$$\pi(\S) = \mathfrak{F}\left[\dot{E}(s)\right]_{s=0}^{s=\S} \tag{5.6}$$

is the general constitutive equation for graphite. In (5.6) π and $\dot{E}$ are symmetric second order tensors. Now if a given material obeying the constitutive relation (5.6) is observed to be transversely isotropic, then the constitutive relation (5.6) must be form invariant under the group of transformations which define the transverse isotropy property. If the z-axis is specified as the symmetry axis of the material, i.e. the h-axis, then the transverse isotropy group is generated by the following symmetry transformations [42],

$$\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.7}$$

along with the identity operation.

Employing an analysis similar to that used by Adkins [43], [44] it can be shown that the Piola stress tensor $\pi(\dot{S})$ can be expressed in terms of a polynomial in the strain rate, where $\pi(\dot{S})$ is invariant under the transversely isotropic group of motions. We can then form symmetric matrix polynomials in the strain rate. We require that these matrix polynomials be invariant under the transverse isotropy group, then by introducing appropriate kernel functions the invariant matrix polynomials may be transformed into our integral approximation [46].

Lianis and DeHoff [45], [46] by applying the theories of Adkins and Pipkin and Rivlin deduced that a symmetric matrix polynomial in the strain rate which is invariant under the transverse isotropy group coincides with the polynomial formed by the irreducible group of products of $\dot{E}(S)$ and $\mathbb{1}$, where

$$\mathbb{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.8}$$

We combine these products with coefficients made up of scalar invariants under the transverse isotropy group. These scalar invariants are polynomials in the elements of an integrity basis for the invariants of $\dot{E}(S)$ under the transversely isotropic group of transformations. The irreducible group of products for any finite number of symmetric 3x3 matrices has been derived by Spencer and Rivlin [47]. The expression for the irreducible integrity basis of scalars for invariants of any number of symmetric 3x3 matrices under the transversely isotropic group of transformations has been derived by Adkins [44].

Following Spencer and Rivlin a symmetric matrix polynomial in two matrices, A and B, can be expressed in terms of the following,

$$\begin{gathered} I, \\ A,\ B, \\ A^2,\ B^2,\ AB + BA, \\ A^2B + BA^2,\ AB^2 + B^2A, \\ A^2B^2 + B^2A^2. \end{gathered} \tag{5.9}$$

In the development of the one dimensional constitutive equation there was no need to consider the transverse isotropy of the graphite specimen in the theory. When considering general three dimensional straining processes, however, the symmetry of the graphite sample must occupy an important place in the theory. The basic results which are needed in order to incorporate the transverse isotropy of graphite into the linear term of the integral approximation have already been laid down. If one finds it desirable to include some of the higher order terms of the integral approximation, then the results of Spencer and Rivlin and Adkins for many 3x3 matrices may be applied. It should be noted that the number of terms which arise in the integrals beyond the triple integral term becomes prohibitively large.

The terms in (5.9) may be applied to the linear term in the integral approximation in order to take the transverse isotropy of the material into account. To this end we set

$$A = \dot{E}(s) \text{ and } B = 1 \tag{5.10}$$

in (5.9) where 1 is defined by (5.8). Since only the linear term of (3.26) is taken into consideration we will, for the sake of consistency, retain only those terms in (5.9) which are linear in $\dot{E}(s)$. Thus with this statement of consistency and (5.10) the table (5.9) reduces to,

$$I, \dot{E}(s), 1, (\dot{E}(s)1 + 1\dot{E}(s)) \tag{5.11}$$

where

$$1 = 1^2 = 1^3 = \dots . \tag{5.12}$$

We now apply Adkins' results on the irreducible group of scalars for symmetric 3x3 matrices which are invariant under the transversely isotropic group of transformations. In keeping with the above statement of consistency, only those terms which are linear in $\dot{E}(s)$ are retained. This yields the irreducible group of scalars for a symmetric 3x3 matrix which are invariant under the group of transversely isotropic motions,

$$\mathrm{Tr}\,\dot{E}(s) \quad \text{and} \quad \mathrm{Tr}\,(1\dot{E}(s)). \tag{5.13}$$

In index notation the linear term in (3.26) may be written as

$$\Pi_{ij} = \int_0^{\mathcal{S}} K_{ijkl}(s)\dot{E}_{kl}(s)\,ds\,. \tag{5.14}$$

The restriction of the integral (5.14) to the description of a transversely isotropic material is now accomplished by combining the terms (5.11) with the coefficients (5.13). Thus the description of the response of any transversely isotropic rate independent material is accomplished by means of the following relation,

$$\int_0^{\mathcal{S}} \Big\{K_1(s)\dot{E}(\mathcal{S}) + K_2(s)[\dot{E}(s)1 + 1\dot{E}(s)] + K_3(s)[\mathrm{Tr}\,\dot{E}(s)]I + K_4(s)[\mathrm{Tr}\,\dot{E}(s)]\,1 + \\ + K_5(s)\,\mathrm{Tr}[1\dot{E}(s)]\,I + K_6(s)\,\mathrm{Tr}[1\dot{E}(s)]\,1\Big\}\,ds\,. \tag{5.15}$$

In (5.15) the $K_i(s)$ are functions of their indicated arguments and the stress and strain rate are second order tensors [46].

The parameter s in (5.14) and (5.15) represents a generalization of the arc length used in the one dimensional representation and it is defined by,

$$s(\tau) = \int_0^{\tau} [\dot{E}_{ij}(\tau')\dot{E}_{ij}(\tau')]^{1/2}\,d\tau'\,. \tag{5.16}$$

It can be seen from (5.16) that the arc length parameter s must be interpreted in terms of the general three dimensional state of strain. Thus the kernel functions $K_i(s)$ in (5.15) are also functions of the three dimensional state of strain. More specifically the kernel functions $K_i(S)$ are functions of the arc length parameter which is an invariant of the strain rate tensor.

C. The Six Dimensional Strain Space

Let us look at some of the properties of the arc length parameter. Consider first the manifold formed by the three dimensional symmetric second order tensors E_{ij}. This manifold forms a six dimensional linear metric space. The metric is defined by means of the scalar product of any two elements $E^{(1)}$ and $E^{(2)}$ and this scalar product is defined by the equation,

$$(E^{(1)}E^{(2)}) = E^{(1)}_{ij}\, E^{(2)}_{ij} \; . \tag{5.17}$$

The norm in this space is formed from the scalar product, viz.,

$$\|E\| = (E_{ij}\, E_{ij})^{1/2} \tag{5.18}$$

and the distance between any two points in the space is given by,

$$\rho(E^{(1)}E^{(2)}) = \sqrt{(E^{(1)}-E^{(2)})(E^{(1)}-E^{(2)})} \; . \tag{5.19}$$

Now let a tensor in this six dimensional space be given as a function of some scalar parameter τ. The derivative with respect to τ

$$\frac{dE_{ij}}{d\tau} \tag{5.20}$$

will not, in general, be a normalized tensor. If, however, this parameter is changed to the new scalar argument, as given by (5.16) then the tensor

$$\frac{dE_{ij}}{ds} = \dot{E}_{ij} = E^{(1)}_{ij} \tag{5.21}$$

will be a normalized tensor [48].

Novozhilov has shown that for an arbitrary tensor curve in our six dimensional space, the following relations hold,

$$\begin{aligned} &\frac{dE^{(1)}}{ds} = q_1 E^{(2)}, \quad \frac{dE^{(2)}}{ds} = -q_1 E^{(1)} + q_2 E^{(3)}, \\ &\frac{dE^{(3)}}{ds} = -q_2 E^{(2)} + q_3 E^{(4)}, \quad \frac{dE^{(4)}}{ds} = -q_3 E^{(3)} + q_4 E^{(5)}, \\ &\frac{dE^{(5)}}{ds} = -q_4 E^{(4)} + q_5 E^{(6)}, \quad \frac{dE^{(6)}}{ds} = -q_5 E^{(5)} . \end{aligned} \tag{5.22}$$

By means of the relations (5.22) it can be seen that the $E^{(i)}$ are normalized and mutually orthogonal.

The applicability of the equations (5.22) can best be seen by means of a two dimensional analog. On a euclidean plane a smooth curve is specified by expressing its position vector $\mathcal{X} = (x_1, x_2)$ as a function if its arc length S. The unit tangent vector $e_1(S)$ and the unit normal vector $e_2(S)$ can be defined if $\mathcal{X}(S)$ is twice continuously differentiable and if the vector $\dot{\mathcal{X}}(S)$ is nowhere zero. Then the vectors $\mathcal{X}(S)$, $e_1(S)$, and $e_2(S)$ are related by the Frenet formulas,

$$\frac{dX}{ds} = e_1 , \quad \frac{de_1}{ds} = q e_2 , \quad \frac{de_2}{ds} = -q e_1 . \tag{5.23}$$

The function $q(s)$ is called the curvature.

Thus for an arbitrary symmetric tensor the relations (5.22) can be thought of as a generalization of the Frenet formulas. By means of these formulas we can determine, for an arbitrary tensor history $E(s)$, a set of six orthonormal tensors at each point of the history.

D. The Straining Program

We wish to obtain results which will be applicable to actual stress-strain curves. To accomplish this a specific straining program must be specified. To this end we assume that the material is strained from its undeformed state and that there is a single strain input to each of the three principal material directions. The assumed straining program valid in the time interval $0 \leq \tau \leq 1/2$ is,

$$\begin{aligned} E_{11}(\tau) &= m_1 \tau \\ E_{22}(\tau) &= -m_2 \tau \\ E_{33}(\tau) &= -m_3 \tau \\ E_{12}(\tau) &= E_{13}(\tau) = E_{23}(\tau) = 0 \quad . \end{aligned} \tag{5.24}$$

In (5.24) we assume that $m_1, m_2,$, and m_3 all have the same sign. With this assumption (5.24) represents a material with one axis in compression and the other two axes in tension, or one axis in tension and the other two axes in compression. We will only consider the case where the material is strained when it is aligned either parallel or perpendicular to its h-axis. This is done so that our theory may be compared to the existing experimental data.

Apparently there is no available published experimental data which will enable us to determine the off diagonal terms of the strain matrix. Thus we are forced to consider only those terms on the main diagonal of the strain matrix. These terms which we call the input to the material are functionally dependent upon the terms on the main diagonal of the stress matrix, or the output. It is evident that for this situation a vector theory could have been developed where the output vector is given as some functional of the input vector. A vector theory would have been easier to develop, but it would have lacked the generality of the present theory. Our theory is capable of describing the shear behavior of graphite and as soon as date of this type is available it can easily be incorporated into our formulation of the three dimensional response of graphite.

By the transverse isotropy property of graphite it is obvious that if the material is strained and it is observed that $m_2 = m_3$, then it can be concluded that the material is being strained parallel to the h-axis.

The arc length parameter S may be determined by applying equation (5.16) to the straining program specified by (5.24). The strain tensor determined by the straining program (5.24) is,

$$E_{ij}(\tau) = \begin{bmatrix} m_1\tau & 0 & 0 \\ 0 & -m_2\tau & 0 \\ 0 & 0 & -m_3\tau \end{bmatrix}. \tag{5.25}$$

The corresponding strain rate tensor is

$$\dot{E}_{ij}(\tau) = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & -m_2 & 0 \\ 0 & 0 & -m_3 \end{bmatrix}. \tag{5.26}$$

The integrand squared of (5.16) may now be written and it is,

$$\dot{E}_{ij}(\tau')\,\dot{E}_{ij}(\tau') = m_1^2 + m_2^2 + m_3^2 . \tag{5.27}$$

Carrying out the operation indicated by (5.16) yields,

$$s(\tau) = M\tau \tag{5.28}$$

where

$$M = (m_1^2 + m_2^2 + m_3^2)^{1/2} . \tag{5.29}$$

The relation (5.28) allows us to write the strain and strain rate tensors, (5.25) and (5.26), in terms of the arc length parameter,

$$E(s) = \begin{bmatrix} m_1 s/M & 0 & 0 \\ 0 & -m_2 s/M & 0 \\ 0 & 0 & -m_3 s/M \end{bmatrix} \tag{5.30}$$

$$\dot{E}(s) = \begin{bmatrix} m_1/M & 0 & 0 \\ 0 & -m_2/M & 0 \\ 0 & 0 & -m_3/M \end{bmatrix} \tag{5.31}$$

The trace of the strain rate tensor (5.31) is,

$$Tr\ \dot{E}(s) = 1/M\ (m_1 - m_2 - m_3)\ . \tag{5.32}$$

The release of strain process corresponding to the straining process (5.25) is

$$\begin{aligned} E_{11}(\tau) &= m_1\left[1 - \tfrac{k_1}{2} + (k_1 - 1)\tau\right] \\ E_{22}(\tau) &= m_2\left[\tfrac{k_2}{2} - 1 + (1 - k_2)\tau\right] \\ E_{33}(\tau) &= m_3\left[\tfrac{k_3}{2} - 1 + (1 - k_3)\tau\right] \\ E_{21}(\tau) &= E_{13}(\tau) = E_{23}(\tau) = 0 \end{aligned} \tag{5.33}$$

where $0 \leq k_1, k_2, k_3 \leq 1$. The restrictions which have been placed on m_1 , m_2 , and m_3 for the straining process are retained for the release of strain process. The equations (5.33) are valid in the time interval $1/2 \leq \tau \leq 1$, and correspond to a simultaneous reversal of all three inputs at the instant in time $\tau = 1/2$. Once again only terms on the main diagonal of the strain matrix are taken into consideration because of the previously given reasons.

At the point of strain reversal we have, as a consequence of either (5.24) or (5.33),

$$E_{11}\,(1/2) = m_1/2$$
$$E_{22}\,(1/2) = -m_2/2 \qquad (5.34)$$
$$E_{33}\,(1/2) = -m_3/2$$

For each of the three inputs there are three relative maximum strains (5.34) to which correspond three permanent sets. In the notation used to describe the permanent set in the scalar case (4.5) we have for $\tau = 1$,

$$E^*_{11} = 1/2\; m_1 k_1$$
$$E^*_{22} = -1/2\; m_2 k_2 \qquad (5.35)$$
$$E^*_{33} = -1/2\; m_3 k_3$$

The strain rate matrix corresponding to the strain matrix defined by (5.33) is

$$\dot{E}_{ij}(\tau) = \begin{bmatrix} m_1(k_1-1) & 0 & 0 \\ 0 & m_2(1-k_2) & 0 \\ 0 & 0 & m_3(1-k_3) \end{bmatrix} \qquad (5.36)$$

and

$$\dot{E}_{ij}(\tau')\dot{E}_{ij}(\tau') = m_1^2(k_1-1)^2 + m_2^2(1-k_2)^2 + m_3^2(1-k_3)^2. \qquad (5.37)$$

Let

$$N = \left[m_1^2(k_1-1)^2 + m_2^2(1-k_2)^2 + m_3(1-k_3)^2\right]^{1/2} \tag{5.38}$$

then

$$s(\tau) = s(\tau)\Big|_{\tau=1/2} + \int_{1/2}^{\tau} N \, d\tau' \quad . \tag{5.39}$$

It can be seen from (5.28) that the first term on the right hand side of (5.39) is equal to $M/2$ and consequently (5.39) becomes

$$s(\tau) = 1/2\ (M - N) + N\tau \tag{5.40}$$

which is valid in the interval $1/2 \leq \tau \leq 1$. Equation (5.40) may be written as

$$\tau = \frac{s}{N} - \frac{1}{2}\left(\frac{M}{N} - 1\right) \tag{5.41}$$

where (5.41) is valid in the arc length interval $M/2 \leq s \leq 1/2(M+N)$. When (5.41) is substituted into (5.33) the strain matrix is obtained as a function of the arc length parameter. By differentiating this strain matrix the following strain rate matrix is obtained,

$$\dot{E}(s) = \begin{bmatrix} m_1/N(k_1-1) & 0 & 0 \\ 0 & m_2/N(1-k_2) & 0 \\ 0 & 0 & m_3/N(1-k_3) \end{bmatrix} . \tag{5.42}$$

The trace of the above strain rate matrix is,

$$\mathrm{Tr}\,\dot{E}(s) = \frac{1}{N}\left[m_1(k_1-1)+m_2(1-k_2)+m_3(1-k_3)\right] . \tag{5.43}$$

Let us now look at the re-straining program for the three dimensional case,

$$\begin{aligned}
E_{11}(\tau) &= m_1(\tau-1+{}^{k_1}/_2) \\
E_{22}(\tau) &= m_2(1-\tau-{}^{k_2}/_2) \\
E_{33}(\tau) &= m_3(1-\tau-{}^{k_3}/_2) \\
E_{12}(\tau) &= E_{13}(\tau) = E_{23}(\tau) = 0
\end{aligned} \tag{5.44}$$

where the previously placed restrictions on m_1, m_2, m_3, k_1, k_2, and k_3 are retained. The strain rate matrix corresponding to the strain matrix defined by (5.44) is,

$$\dot{E}_{ij}(\tau) = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & -m_2 & 0 \\ 0 & 0 & -m_3 \end{bmatrix} \tag{5.45}$$

Thus in the previously defined notation

$$\dot{E}_{ij}(\tau')\,\dot{E}_{ij}(\tau') = M^2 \tag{5.46}$$

By means of (5.40) evaluated at $\tau = 1$ and the relation (5.46), the arc length parameter can be determined in the time interval $\tau = 1$,

$$s(\tau) = \tfrac{1}{2}(M+N) + \int_1^{\tau} M \, d\tau'$$

$$= \tfrac{1}{2}(N-M) + M\tau \, . \qquad (5.47)$$

Equation (5.48) may be written as

$$\tau = \frac{1}{M}\left[s + \tfrac{1}{2}(M-N)\right] \qquad (5.48)$$

and by using this the strain rate matrix may be written in terms of the arc length parameter,

$$\dot{E}_{ij}(s) = \begin{bmatrix} m_1/M & 0 & 0 \\ 0 & -m_2/M & 0 \\ 0 & 0 & -m_3/M \end{bmatrix} . \qquad (5.49)$$

The trace of this matrix is,

$$\mathrm{Tr}\,\dot{E}(s) = \frac{1}{M}(m_1 - m_2 - m_3) \quad . \qquad (5.50)$$

The relations (5.49) and (5.50) are valid in the interval $\tau \geq 1$ or $s \geq \frac{1}{2}(M+N)$.

E. Discussion of the Kernel Functions

At this point only the six constants m_1, m_2, m_3, E_1^*, E_2^*, E_3^*, must be specified in order to apply our stress-strain relations for straining, release of strain, and re-straining. These constants arise from the specification of the inputs to the material. Other constants will appear in our final formulation and they will be associated with the kernel functions $K_i(s)$. We now provide further reasons for choosing the arc length parameter as our measure of the deformation and also explain how the kernel functions are to be chosen.

We have previously endeavored to show, by comparison with the classical theory of plasticity, that the arc length parameter is a suitable measure of the amount of deformation which has taken place within the material. Bridgeman has shown experimentally that hydrostatic pressures do not cause any appreciable plastic deformation in metals and the plastic deformations have been shown to take place along shear planes. Thus in most theories of plastic deformation only the deviatoric stresses are of any import.

Theories of plastic deformation exist which take normal stresses into account. Some theories of granular work hardening materials are of this type. In fact, in the theory of the plastic cavitation of granular materials it has been shown that the residual increase in volume is not proportional to the work done in the deformation, but to the arc length of the plastic deformation path [49]. Here the parameter which seems to provide a natural description of the cyclic loading behavior of granular materials is the arc length parameter.

Graphite may be classified as a granular material. Any polycrystalline material which is both microscopically and supermicroscopically heterogeneous and anisotropic (on account of the granular structure and the individual defects in the structure of each grain), forms a statically indeterminate system from the viewpoint of structural mechanics. As the loading progresses the elements in this system gradually begin to deform plastically. This is observed macroscopically as the monotonic increase in the coefficient of friction. While these plastic deformations progress elastic interactions are set up between the elements of the system and this is interpreted as the hardening of the material. Thus we see that since graphite exhibits the above properties it may be classified as a granular material.

In some of the theories of granular materials the arc length parameter is found to be a suitable parameter for describing the mechanical response of the materials considered. It has already been noted that the residual increase in volume, in a granular material, due to cyclic loading has been proven to be proportional to the arc length of the plastic deformation path. Thus the arc length should also be a suitable parameter for describing the mechanical behavior of polycrystalline graphite.

This discussion of the theory of the plastic cavitation of a granular material and the previous discussions of work hardening plasticity tend to indicate that the arc length should be the natural parameter to use in describing the mechanical behavior of graphite. There are many other examples pertinent to the use of an arc length, or work hardening parameter in classical plasticity and related topics, which have appeared since Odqvist's work [50] which interpreted the plastic yield condition in terms of streamlines.

As in some of the theories of granular materials the mechanical response of graphite does not seem to lend itself to description by the deviatoric stress or strains. We have assumed that the normal stresses also contribute to the plastic deformation in graphite. While it is true that a single crystal of graphite will not be influenced, to a great extent, by hydrostatic stresses, the macroscopic problem of a, to a large degree, randomly oriented polycrystalline structure will be dependent upon the magnitude of these hydrostatic stresses.

Hence from both our geometric and physical arguments it appears that the arc length parameter is indeed a reasonable measure of the deformation of polycrystalline graphite.

The kernel functions $K_i(s)$ have been specified as functions of the arc length parameter which is dependent upon the entire three dimensional state of strain. From our experience with the one dimensional representation of graphite it is reasonable to assume that the $K_i(s)$ are of exponential form.

In fact, with the choice of the arc length as the intrinsic time associated with graphite the exponential form of the kernel function is necessary in order that the three dimensional theory reduce to the previously derived one dimensional model. That is, when there is only one strain input the three dimensional theory should yield the previously derived one dimensional model.

To this end we assume that the kernel functions are of the following form, for monotonic or cyclic straining

$$k_i(s) = c_i \cdot \exp\left\{\left[a + bn\left(\frac{\mathfrak{s}^*}{\hat{\mathfrak{s}}} - 1\right)\right]\hat{\mathfrak{s}} - bs\right\} \quad (i = 1, 2, \ldots, 6) \quad . \tag{5.51}$$

The relation (5.15) has the same basic form as (4.24), but (5.15) is of a different character. In (5.15) the parameter S is no longer dependent upon a scalar strain, but upon the entire three dimensional state of strain. While we have continued to use the letter S to denote both of these parameters, in the one case S is a path length in a one dimensional space, and in the other S is a path length in a six dimensional space.

The value of the arc length corresponding to the reversal of strain is again denoted by $\hat{\mathfrak{s}}$, and $\mathfrak{s}^*$ is the arc length corresponding to the unloaded condition. The value of $\mathfrak{s}^*$ corresponding to the permanent set defined by (5.35) is

$$\mathfrak{s}^* = \tfrac{1}{2}(M + N) \quad . \tag{5.52}$$

For this case of three dimensional strain the magnitude of the strain is given by the scalar formed from the square root of the scalar product,

$$\|E\| = \sqrt{E_{ij}E_{ij}} \tag{5.53}$$

This is also the norm of the six dimensional space formed by the components of the strain tensor. The straining process we are considering is well defined insofar as all three strain inputs are reversed at the same time, and it is because of this that there is no problem in specifying complete strain reversals. Hence we may specify n as the number of complete strain reversals. a, b, and the c_i denote constants which are to be determined. In this manner the three dimensional formalism we have developed will reduce to the previously derived one dimensional case.

F. Application of the Kernel Functions

The kernel functions (5.51) are now substituted into the constitutive equation (5.15) while making use of the relations (5.30), (5.31), and (5.32). Thus the relation for straining in the interval $0 \leq \gamma \leq \frac{1}{2}$, $0 \leq s \leq \frac{M}{2}$, may be written as

$$\pi(\varphi) = \frac{1}{M}\int_0^{\varphi} \left\{ c_1 \begin{bmatrix} m_1 & 0 & 0 \\ 0 & -m_2 & 0 \\ 0 & 0 & -m_3 \end{bmatrix} + (2C_2 + C_6) \begin{bmatrix} m_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right.$$

$$+ c_3 \begin{bmatrix} m_1 - m_2 - m_3 & 0 & 0 \\ 0 & m_1 - m_2 - m_3 & 0 \\ 0 & 0 & m_1 - m_2 - m_3 \end{bmatrix} + c_4 \begin{bmatrix} m_1 - m_2 - m_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.54}$$

$$\left. + c_5 \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_1 \end{bmatrix} \right\} \exp\left\{ \left[aM/2 + bN/2 \right] \hat{\varphi} - bs \right\} ds \; .$$

Thus we may write,

$$\pi_{11}(\varphi) = \frac{1}{Mb}\left\{ \left[c_1 + 2c_2 + c_5 + c_6 \right] m_1 + \left[c_3 + c_4 \right](m_1 - m_2 - m_3) \right\}\left[1 - e^{-b\varphi} \right], \tag{5.55}$$

$$\Pi_{22}(\S) = \frac{1}{Mb}\left\{c_3(m_1-m_2-m_3)+c_5 m_1 - c_1 m_2\right\}\left[1-e^{-b\S}\right], \qquad (5.56)$$

$$\Pi_{33}(\S) = \frac{1}{Mb}\left\{c_3(m_1-m_2-m_3)+c_5 m_1 - c_1 m_3\right\}\left[1-e^{-b\S}\right], \qquad (5.57)$$

because $\hat{\S} = 0$.

For the straining process in the 11-direction we have,

$$E_{11}(s) = \frac{m_1}{M} s \qquad (5.58)$$

and the first derivative of the stress in the 11-direction is,

$$\frac{d\Pi_{11}}{dE_{11}} = \frac{1}{m_1}\left\{\left[c_1+2c_2+c_5+c_6\right]m_1+\left[c_3+c_4\right](m_1-m_2-m_3)\right\} e^{-b\S}. \qquad (5.59)$$

For graphite the first derivative (5.59) must always be greater than zero (assuming that $m_1 > 0$) for all $\S$. In particular when $\S = 0$ we have,

$$\left.\frac{d\Pi_{11}}{dE_{11}}\right|_{E_{11}=0} = \frac{1}{m_1}\left\{\left[c_1+2c_2+c_5+c_6\right]m_1+\left[c_3+c_4\right](m_1-m_2-m_3)\right\} > 0. \qquad (5.60)$$

The constant on the left hand side of the inequality (5.60) may be interpreted as the tangent modulus of the graphite specimen for the stress-strain curve in the 11-direction.

From the condition that the second derivative of the stress

$$\frac{d^2\Pi}{dE_{11}^2} = \frac{-bM}{m_1^2}\left\{\left[c_1+2c_2+c_5+c_6\right]m_1+\left[c_3+c_4\right](m_1-m_2-m_3)\right\} e^{-b\S} \qquad (5.61)$$

always be less than zero we obtain the condition that

$$b > 0$$

The relation between the applied strains and the stresses for the release of strain process, in the interval $\frac{1}{2} \leq \gamma \leq 1$ or equivalently $\frac{M}{2} \leq S \leq \frac{1}{2}(M+N)$, is by (5.15), (5.42), (5.43), and (5.51),

$$\Pi(S) = \Pi(\varsigma)\Big|_{\varsigma=\frac{M}{2}} + \frac{1}{N}\int_{\frac{M}{2}}^{S}\Bigg\{C_1\begin{bmatrix} m_1(k_1-1) & 0 & 0 \\ 0 & m_2(1-k_2) & 0 \\ 0 & 0 & m_3(1-k_3)\end{bmatrix} +$$

$$+(2c_2+c_6)\begin{bmatrix} m_1(k_1-1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix} + c_3\left[m_1(k_1-1)+m_2(1-k_2)+m_3(1-k_3)\right]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix} + \tag{5.62}$$

$$+C_4\begin{bmatrix} m_1(k_1-1)+m_2(1-k_2)+m_3(1-k_3) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix} + c_5 m_1(k_1-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}\Bigg\}\exp\left\{\frac{aM}{2}+\frac{bN}{2}-b\varsigma\right\}ds.$$

where we have set $n = 1$ and $M/2$. Thus we may write

$$\Pi_{11}(\varsigma) = \Pi_{11}(\varsigma)\Big|_{\frac{M}{2}} + \frac{1}{bN}\Big\{\left[c_1+2c_2+c_5+c_6\right]m_1(k_1-1) + \left[c_3+c_4\right]\left[m_1(k_1-1) +m_2(1-k_2)+m_3(1-k_3)\right]\exp\left\{\frac{aM}{2}+\frac{bN}{2}\right\}\left[\exp\left(-\frac{bM}{2}\right)-\exp(-b\varsigma)\right] \tag{5.63}$$

$$\Pi_{22}(\varsigma) = \Pi_{22}(\varsigma)\Big|_{\varsigma=\frac{M}{2}} + \frac{1}{bN}\Big\{c_1 m_3(1-k_3) + c_3\left[m_1(k_1-1)+m_2(1-k_2)+m_3(1-k_3)\right] + c_5 m_1(k_1-1)\Big\}\exp\left\{\frac{aM}{2}+\frac{bN}{2}\right\}\left[\exp\left(-\frac{bM}{2}\right)-\exp(-b\varsigma)\right] \tag{5.64}$$

$$\Pi_{33}(\xi)=\Pi_{33}(\xi)\Big|_{\xi=\frac{M}{2}}+c_1m_3(1-k_3)+c_3[m_1(k_1-1)+m_2(1-k_2)+ m_3(1-k_3)]+c_5m_1(k_1-1)]\exp\left\{\frac{aM}{2}+\frac{bN}{2}\right\} \left[\exp(-\frac{bM}{2})-\exp(-b\xi)\right]. \tag{5.65}$$

As in the one dimensional case let us require that, in the 11-direction, the stress return to zero after the completion of the release of strain process. Hence,

$$\Pi_{11}(\frac{M+N}{2})=\frac{1}{Mb}\{[c_1+2c_2+c_5+c_6]m_1+[c_3+c_4](m_1-m_2-m_3)\}[1-\exp(-\frac{bM}{2})] +\frac{1}{bN}\{[c_1+2c_2+c_5+c_6]m_1(k_1-1)+[c_3+c_4][m_1(k_1-1)+m_2(1-k_2)+m_3(1-k_3)]\} \exp\left\{\frac{aM}{2}+\frac{bN}{2}\right\}\left\{\exp(-\frac{bM}{2})-\exp\left[-\frac{b}{2}(M+N)\right]\right\}=0 \tag{5.66}$$

By rearranging terms we obtain,

$$[c_1+2c_2+c_5+c_6]\left\{\frac{1}{Mb}\left[m_1(1-\exp(-\frac{bM}{2}))\right]\right\}+\frac{1}{bN}\left\{[m_1(k_1-1)]\cdot \cdot\exp\left\{\frac{aM}{2}+\frac{bN}{2}\right\}\left\{\exp(-\frac{bM}{2})-\exp\left[-\frac{b}{2}(M+N)\right]\right\}\right\}+ +[c_3+c_4]\frac{1}{bM}[m_1-m_2-m_3][1-\exp(-\frac{bM}{2})]+\frac{1}{bN}[m_1(k_1-1)+m_2(1-k_2)+ +m_3(1-k_3)]\exp\left\{\frac{aM}{2}+\frac{bN}{2}\right\}\left\{\exp(-\frac{bM}{2})-\exp\left[-\frac{b}{2}(M+N)\right]\right\}=0 \tag{5.67}$$

or

$$[c_1+2c_2+c_5+c_6]\left[\left\{[1-\exp(-\frac{bM}{2})]\frac{m_1}{Mb}\right\}+\frac{1}{bN}[m_1(k_1-1)]\exp\left\{\frac{aM}{2}+\frac{bN}{2}\right\} [1-\exp(-\frac{bN}{2})]\right]+[c_3+c_4]\frac{1}{bM}(m_1-m_2-m_3)[1-\exp(-\frac{bM}{2})]+\frac{1}{bN}[m_1(k_1-1)+ +m_2(1-k_2)+m_3(1-k_3)]\exp\left\{\frac{aM}{2}+\frac{bN}{2}\right\}[1-\exp(-\frac{bN}{2})]=0 \tag{5.68}$$

The above equation will be given values of $\mathcal{M}$, $\mathcal{N}$, k_i, m_i, b, and c_i determine the material parameter a.

Let us now look at the re-straining process,

$$\Pi(\xi)=\Pi(\xi)\Big|_{\xi=\frac{1}{2}(\mathcal{M}+\mathcal{N})}+\frac{1}{\mathcal{M}}\int_{\frac{1}{2}(\mathcal{M}+}^{\xi}\left\{\begin{bmatrix} m_1 & 0 & 0\\ 0 & -m_2 & 0\\ 0 & 0 & -m_3\end{bmatrix}+2c_2\begin{bmatrix} m_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\right.$$

$$+c_3(m_1-m_2-m_3)\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+c_4\begin{bmatrix} m_1-m_2-m_3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+c_5\begin{bmatrix} m_1 & 0 & 0\\ 0 & m_1 & 0\\ 0 & 0 & m_1\end{bmatrix}+ \qquad (5.69)$$

$$\left.+c_6\begin{bmatrix} m_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right\}\exp\left\{\left[\frac{a\mathcal{M}}{2}+b\mathcal{N}\right]-bs\right\}ds.$$

The first term on the right side of (5.69) is zero, because we have assumed that the strain release process returns the graphite specimen to a state of zero stress. Thus we may write,

$$\Pi_{11}(\xi)=\frac{1}{\mathcal{M}b}\left\{\left[c_1+2c_2+c_5+c_6\right]m_1+\left[c_3+c_4\right](m_1-m_2-m_3)\right\}\cdot$$
$$\cdot\exp\left[\frac{a\mathcal{M}}{2}+b\mathcal{N}\right]\left\{\exp\left[\frac{b}{2}(\mathcal{M}+\mathcal{N})\right]-\exp(-b\xi)\right\}, \qquad (5.70)$$

$$\Pi_{22}(\xi)=\frac{1}{\mathcal{M}b}\left\{-c_1m_2+c_3(m_1-m_2-m_3)+c_5m_1\right\}\cdot$$
$$\cdot\exp\left[\frac{a\mathcal{M}}{2}+b\mathcal{N}\right]\left\{\exp\left[-\frac{b}{2}(\mathcal{M}+\mathcal{N})\right]-\exp(-b\xi)\right\}, \qquad (5.71)$$

$$\Pi_{33}(\xi)=\frac{1}{Mb}\left\{-c_1m_3+c_5(m_1-m_2-m_3)+c_5m_1\right\}\cdot$$
$$\cdot\exp\left[\frac{aM}{2}+bN\right]\left\{\exp\left[-\frac{b}{2}(M+N)\right]-\exp(-b\xi)\right\}. \tag{5.72}$$

The general triaxial equations for straining, strain release and restraining are now complete. In each case the three dimensional equations will reduce to the previously derived one dimensional equations when only one input is considered.

F. Concerning the Applicability of the Results

In order to better see the applicability of the results let us look at the experiments we have been using as a check of our theory. In these experiments a stress input was applied to a graphite sample in one direction only and the strains, being the outputs, were observed along the principle axes of the material. Thus a stress was applied to only one principal direction, the other two directions having no stresses applied to them. We then reason that if, in our theory, the observed strains are used as the inputs to the material, then the theory should predict the correct experimental stresses which were applied to the material. This requires that we should predict a stress in the 11-direction, but no stresses in the 22- and 33-directions.

Thus we see that in order to apply the existing date we will have to assume the type of data reversibility explained above. Specifically, we must assume that the stress input-strain output data can be used in our strain input-stress output formalism. We do not know of any experimental evidence which would tend to substantiate this type of reversibility of data, but we know of no experimental evidence which would tend to contradict this type of behavior in graphite. For the moment, however, let us assume that this is a valid type of behavior for graphite.

The question arises: why not invert the constitutive equation and obtain a relation with a stress input and strain output? In this manner we would be able to handle stress inputs and apply the experimental data directly. There are, however, difficulties in using this method. The difficulties lie not in the mechanics of the inversion process, but in the physical significance of the arc length of the stress path in stress space. An additional difficulty arises in finding suitable kernel functions to be used in this representation. In the stress input formalism the kernels do not necessarily have to be related to the kernels in our theory. Even more, they do not necessarily have to be of exponential form. Also, if we perform this inversion all of our arguments based upon the concept of work hardening plasticity will no longer have any applicability, and these arguments could not be carried over to apply to the arc length in stress space, since one does not usually speak of a stress hardening parameter. The arc length as a function of the stresses does not have the same physical significance as the arc length in strain space. To provide the justification for the use of the arc length in stress space as the measure of the deformation would require a great deal of effort.

At this point let us look at the consequences of the assumption of reversibility of the data. First this requires that the stresses in the 22- and 33-directions be zero for the straining, release of strain, and re-straining processes. This can be accomplished either by requiring that C_1, C_3, and C_5 vanish, or by finding suitable combinations of the m_i, k_i, and C_1, C_3, and C_5.

Thus we have a reasonable representation for the behavior of graphite for the experimental case discussed. One must not forget, however, that we have assumed that we can take stress input-strain output data and use it as if it were strain input-stress output data. Errors will probably arise because of this. This is not the only source of error in our development.

Any discrepancy between theory and experiment may be due to our use of the linear term in the integral approximation. In applying the integral approximation (3.26) to the basic constitutive equation, the first term of the approximation was taken to represent the mechanical response of graphite, while the remaining terms were neglected. It might very well be that this single term of the approximation is not sufficient to describe the mechanical response of reactor grade graphite. Engineers have found that the linear term of the integral approximation is not always sufficient to describe the mechanical behavior of some materials. For Example, Wang and Onat [51] have shown that even the first few terms of the integral approximation do not yield results which can predict, with reasonable accuracy, the mechanical behavior of 1100 aluminum at 300° F.

This situation may also exist for the general three dimensional response of some types of reactor grade graphite. If so, then perhaps there is an argument, strong enough to outweigh arguments of complexity and unwieldiness, to include higher order terms of the approximation in order to obtain a better representation of the mechanical properties of graphite. However, as Pipkin has pointed out [52] there is no guarantee that the addition of more terms of the approximation will produce a better approximation. In fact, the addition of just a few higher order terms may yield an approximation which is less accurate than that obtained with the linear term alone. Thus if higher order terms are needed we cannot say how many more terms will suffice in order to get the job done. Also, there is no theory of convergence which we can apply to our integral series.

H. Comparison to the Experimental Data

Consider a specimen of graphite exhibiting one weak axis and two strong axes. The weak axis is in the 11-direction. Thus all data taken when the material was loaded parallel to the 11-direction will show that the data for the 22- and 33-directions are the same. If the specimen of graphite is loaded parallel to the 22-direction then we will obtain three different strain outputs. By reversing this data we find that we should have three inputs to the material and two zero outputs. The following data is the result of loading a graphite specimen parallel to its 22-axis. The relative maximums are determined by,

$$\left.\begin{aligned} m_1 &= 4{,}200 \\ m_2 &= 25{,}500 \\ m_3 &= 3{,}500 \end{aligned}\right\} \qquad (5.73)$$

and the permanent set by

$$\left.\begin{aligned} k_1 &= 0.143 \\ k_2 &= 0.134 \\ k_3 &= 0.145 \end{aligned}\right\} \tag{5.74}$$

using these values we find that

$$\mathcal{M} = 26,080 \tag{5.75}$$

$$\mathcal{N} = 22,573 \tag{5.76}$$

$$\hat{\mathcal{S}} - \mathcal{S}^* = 11,286 \tag{5.77}$$

The process defined by the above constants will be referred to as case I.

When the material was strained parallel to the II-direction, the following constants were obtained,

$$\left.\begin{aligned} m_1 &= 49,600 \\ m_2 &= 5,100 \\ m_3 &= 5,100 \end{aligned}\right\} \tag{5.78}$$

and

$$\begin{aligned} k_1 &= 0.130 \\ k_2 &= 0.118 \\ k_3 &= 0.118 \end{aligned} \tag{5.79}$$

We find that,

$$\mathcal{M} = 50,120 \tag{5.80}$$

$$\mathcal{N} = 43,617 \tag{5.81}$$

$$\hat{\mathcal{S}} - \mathcal{S}^* = 21,808 \tag{5.82}$$

The constants (5.78)-(5.82) will serve to define the process we will refer to as case II. Straining parallel to the 33-direction is equivalent to straining parallel to the 22-direction and thus there is no need to consider this process separately.

For case I we have the following basic equations for straining,

$$\Pi_{11}(\xi)=\frac{1}{M}b\left\{-[c_1+2c_2+c_5+c_6]m_1+[c_3+c_4](-m_1+m_2+m_3)\right\}\cdot$$
$$\cdot\left[1-\exp(-b\xi)\right] \tag{5.83}$$

where

$$E_{11} = -m_1 \tau \tag{5.84}$$
$$\xi = M\tau \tag{5.85}$$

$$\Pi_{22}(\xi)=\frac{1}{M}b\left\{c_1m_2+c_3(-m_1+m_2+m_3)-c_5m_1\right\}\cdot$$
$$\cdot\left[1-\exp(-b\xi)\right] \tag{5.86}$$

where

$$E_{22} = m_2 \tau \tag{5.87}$$

$$\Pi_{33}(\xi)=\frac{1}{M}b\left\{-c_1m_3+c_3(-m_1+m_2+m_3)-c_5m_1\right\}\cdot$$
$$\cdot\left[1-\exp(-b\xi)\right] \tag{5.88}$$

where

$$E_{33} = -m_3 \tau \tag{5.89}$$

Equations (5.83)-(5.89) are valid for

For the release of strain in case I, the basic equations are,

$$\Pi_{11}(\S) = \Pi_{11}(13{,}040) + \frac{1}{Nb}\{[c_1 + 2c_2 + c_5 + c_6]\, m_1(1-k_1) + \\ + [c_3 + c_4][m_1(1-k_1) + m_2(k_2-1) + m_3(1-k_3)]\} \cdot \\ \cdot \exp\left[\frac{aM}{2} + bN\right]\left[\exp\left(-\frac{bM}{2}\right) - \exp(-b\S)\right] \tag{5.90}$$

where

$$E_{11} = m_1\left[\frac{k_1}{2} - 1 + (1-k_1)\tau\right] \tag{5.91}$$

$$\S = \frac{1}{2}(M-N) + N\tau \tag{5.92}$$

$$\Pi_{22}(\S) = \Pi_{22}(13{,}040) + \frac{1}{Nb}\{c_1 m_2(k_2-1) + \\ + c_3[m_1(1-k_1) + m_2(k_2-1) + m_3(1-k_3)] + c_5 m_1(1-k_1)\} \cdot \\ \cdot \exp\left[\frac{aM}{2} + bN\right]\left[\exp\left(\frac{-bM}{2}\right) - \exp(-b\S)\right] \tag{5.93}$$

where

$$E_{22} = m_2\left[1 - \frac{k_2}{2} + (1-k_2)\tau\right] \tag{5.94}$$

$$\Pi_{33}(\S) = \Pi_{33}(13{,}040) + \frac{1}{Nb}\{c_1 m_3(1-k_3) + \\ + c_3[m_1(1-k_1) + m_2(k_2-1) + m_3(1-k_3)] + c_5 m_1(1-k_1)\} \cdot \\ \cdot \exp\left[\frac{aM}{2} + bN\right]\left[\exp\left(\frac{-bM}{2}\right) - \exp(-b\S)\right] \tag{5.95}$$

where

$$E_{33} = m_3\left[k_3/2 - 1 + (1-k_3)\tau\right] \; . \tag{5.96}$$

Equations (5.90)-(5.96) are valid for $1/2 \le \tau \le 1, \; \frac{M}{2} \le \S \le \frac{1}{2}(M+N)$

For restraining in case I, the basic equations are,

$$\Pi_{11}(\S)+\frac{1}{M}b\left\{-[c_1+2c_2+c_5+c_6]m_1+[c_3+c_4](-m_1+m_2+m_3)\right\}\cdot$$
$$\cdot\exp\left[\frac{aM}{2}+bN\right]\left\{\exp\left[-\frac{b}{2}(M+N)\right]-\exp(-b\S)\right\} \tag{5.97}$$

where

$$E_{11}=m_1(1-\tau-\frac{k_1}{2}) \tag{5.98}$$
$$\S=\tfrac{1}{2}(N-M)+M\tau \tag{5.99}$$
$$\Pi_{22}(\S)=\frac{1}{M}b\left\{c_1m_2+c_3(-m_1+m_2+m_3)-c_5m_1\right\}\cdot$$
$$\cdot\exp\left[\frac{aM}{2}+bN\right]\left\{\exp\left[-\frac{b}{2}(M+N)\right]-\exp(-b\S)\right\} \tag{5.100}$$

where

$$E_{22}=m_2(\tau-1+\frac{k_2}{2}) \tag{5.101}$$
$$\Pi_{33}(\S)=\frac{1}{M}b\left\{-c_1m_3+c_3(-m_1+m_2+m_3)-c_5m_1\right\}\cdot$$
$$\exp\left[\frac{aM}{2}+bN\right]\left\{\exp\left[-\frac{k}{2}(M+N)\right]-\exp(-b\S)\right\} \tag{5.102}$$

where

$$E_{33}=m_3(1-\tau-k_3/2)\ . \tag{5.103}$$

Equations (5.97)-(5.103) are valid for

$$\tau\geq 1\ ,\qquad \S\geq\tfrac{1}{2}(M+N).$$

The basic equations for case II have already been stated and they are (5.24), (5.28), (5.33), (5.40), (5.44), (5.55), (5.56), (5.57), (5.63), (5.64), (5.65), (5.70), (5.71), and (7.73). In all these equations we must set $m_2=m_3$.

In case I the stresses in the 33- and 11-directions should be zero for straining, the release of strain, and re-straining. Thus (5.83) reduces to

$$-4200\,C_1 + 17{,}800\,C_2 = 0 \tag{5.104}$$

and (5.90) reduces to

$$3599\,C_1 - 15{,}492\,C_2 = 0 \tag{5.105}$$

where

$$C_1 = c_1 + 2c_2 + c_5 + c_6 \tag{5.106}$$

$$C_2 = c_3 + c_4 \quad . \tag{5.107}$$

Equations (5.104) and (5.105) are essentially the same relation to within experimental error. Constants which satisfy (5.104) will very nearly satisfy (5.105). These conditions obtained from the re-straining equations are the same as those obtained from the straining equations.

From equation (5.88) we obtain the condition

$$-3{,}500c_1 + 17{,}800\,c_3 - 4{,}200c_1 = 0 \tag{5.108}$$

and from equation (5.95) we obtain

$$2{,}992\,c_1 - 15{,}492\,c_3 + 3{,}599\,c_5 = 0 \tag{5.109}$$

We also consider (5.108) and (5.109) as essentially the same relation to within experimental error.

Similarly in case II from Equation (5.57) we obtain the independent equation

$$-5{,}100C_1 + 39{,}400C_3 + 49{,}600C_5 = 0 \tag{5.110}$$

We set

$$b = 0.00015 \tag{5.111}$$

as the best fit to the monotonic straining portion of the stress-strain curve of case I. When $\S = M/2$ we see that $\pi_{22}(\S) = 3600$ lbs. in case I and $\pi_{11}(\S) = 3600$ lbs. in case II. Thus we obtain the additional conditions

$$25{,}500C_1 + 17{,}800C_3 - 4{,}200C_5 = 163.64 \tag{5.112}$$

and

$$49{,}600\,C_1 + 39{,}400\,C_2 = 13{,}859 \tag{5.113}$$

Equation (5.112) comes from (5.86) and equation (5.113) comes from (5.55).

Solving equations (5.108), (5.109), and (5.112) simultaneously yields

$$C_1 = 0.564 \tag{5.114}$$

$$C_3 = 0.105 \tag{5.115}$$

$$C_5 = -0.0254 \quad . \tag{5.116}$$

Equations (5.113) and (5.104) yield

$$C_1 = 0.470 \tag{5.117}$$

$$C_2 = 0.111\,. \tag{5.118}$$

We can now solve for c_4 which is found to be

$$c_4 = 0.006 \tag{5.119}$$

Substituting the appropriate values into (5.86) yields

$$\Pi_{22}(\S) = 4{,}194\left[1 - \exp(0.00015\,\S)\right] \tag{5.120}$$

from which we obtain the values in table IV, which describe the straining process.

TABLE IV

τ	$\S$	$\Pi(\S)$ lbs.	E μin/in
0.0	0.000	0.000	0.000
0.1	2,608	1,358	2,550
0.2	5,216	2,276	5,100
0.3	7,824	2,897	7,650
0.4	10,432	3,317	10,200
0.5	13,040	3,600	12,750

The constant a is found by requiring that the stress return to zero when the strain is released to the value of the permanent set. In this manner we find that

$$a = 0.00002462 \tag{5.121}$$

The equation for the release of strain (5.93) becomes

$$\pi_{22}(\S) = -848 + 31{,}458 \exp(-0.00015\S) \tag{5.122}$$

from which we obtain the values in table V.

TABLE V

τ	$\S$	$\pi(\S)$ lbs.	E μin/in
0.6	15,298	2.323	10,542
0.7	17,555	1.421	8,334
0.8	19,812	763	6,126
0.9	22,070	300	3,917
1.0	24,327	-29	1,709

For restraining we substitute the appropriate values into (5.100) and obtain

$$4{,}420 - 170{,}000 \exp(-b\S) \tag{5.123}$$

for which we obtain the values shown in table VI.

TABLE VI

τ	ξ	$\pi(\xi)$ lbs.	E μ in/in
1.1	26,934	1,428	4,258
1.2	29,542	2,397	6,808
1.3	32,150	3,052	9,358
1.4	34,758	3,495	11,908
1.5	37,366	3,794	14,458

The values of the stress in the 11- and 33- directions are less than 0.50 lbs. when the strain is 13,040 μin/in.

Let us now consider case II. From the equation for straining (5.55) we obtain (5.124) which upon the substitution of the appropriate values for the constants yields

$$\pi_{11}(\xi) = 3{,}683\,[1-\exp(-0.00015\,\xi)]\,. \qquad (5.124)$$

Table VII shows the values obtained from (5.124).

TABLE VII

τ	ξ	$\pi(\xi)$ lbs.	E μ in/in
0.0	0.000	0.000	0.000
0.1	5,012	1,946	4,960
0.2	10,024	2,864	9,920
0.3	15,036	3,296	14,880
0.4	20,048	3,500	19,840
0.5	25,060	3,597	24,800

Equation (5.63) yields the following relation for the release of strain when the appropriate values are substituted,

$$\Pi_{11}(\S) = -128.0 + 160{,}000 \exp(-0.00015\S). \qquad (5.125)$$

In (5.125) we have used the following value for a,

$$a = 0.0000198 \qquad (5.126)$$

which was obtained by requiring that the stress return to zero when the strain attains the value of the permanent set. Table VIII is obtained from equation (5.125).

TABLE VIII

τ	$\S$	$\Pi(\S)$ lbs.	E μin/in
0.6	29,422	1,804	20,386
0.7	33,784	880	16,071
0.8	38,146	396	11,755
0.9	42,102	161	7,440
1.0	46,419	23	3,125

For the re-straining process, equation (5.70) with the appropriate constants yields the following relation,

$$\Pi_{11}(\S) = 3{,}690 - 4{,}169{,}500 \exp(-0.00015\S). \qquad (5.127)$$

From this equation Table IX is obtained.

TABLE IX

τ	ξ	$\pi(\xi)$ lbs.	ϵ μin/in
1.1	51,880	1,950	8,184
1.2	56,892	2,870	13,144
1.3	61,904	3,303	18,104
1.4	66,916	3,507	23,064
1.5	71,928	3,604	28,024

The curve obtained from tables IV, V, and VI is illustrated in Fig. 14, and the curve obtained from tables VII, VIII, and IX is illustrated in Fig. 15.

Thus we see that the theoretical and the experimental curves are in agreement in case I even though a combination of the date for the 22- and 33- directions were used in the theory. More specifically, the data for the 22- and 33- directions should have been identical as a consequence of the transverse isotropy property, but this was not the actual case as can be seen by comparing the experimental curves in Fig. 14a and Fig. 14b.

In case II the fit is not very good, however, the strains are extremely large and this may account for the discrepancy between theory and experiment. When applying the integral approximations to such large strains it has often been found that the linear term of the approximation is not sufficient to closely describe the mechanical behavior of the considered material when the strains are very large. With this in mind our theoretical curve seems more reasonable. It is felt that the consideration of the third order term of the integral approximation will yield a theoretical curve which more closely approximates the experimental curve.

Note that we have successfully obtained the result of having negligible stresses in the directions to which there was no stress applied in the actual experiment. This combined with the results for the 22-direction in case I and 11-direction in case II constitutes a reasonable model of the three dimensional mechanical behavior of polycrystalline reactor grade graphite.

VI. CONCLUSIONS AND RECOMMENDATIONS

The applicability of our three dimensional theory is seriously impared by a lack of knowledge of the form of the off-diagonal terms of the strain matrix. The theory itself remains a valid one, but its applicability remains small. This can be remedied as soon as the needed data can be supplied. All that is needed is some truly three dimensional data, or some simple triaxial data which can be obtained from various general strain input-stress output experiments. The only data available to us implies that we have zero outputs in two directions, and this is not a true test of the theory. This data does not even allow the determination of all the constants.

Our one dimensional theory is, along with Woolley's work, the best description of the one dimensional mechanical behavior of graphite. It is also the best description of the one dimensional cyclic straining behavior of graphite to be found in the literature. The only other description of this type of cyclic straining is that of Jenkins and as mentioned in the first chapter it has severe limitations.

Our three dimensional theory is the only one of its type which has been applied to graphite, and in this respect is completely original. It is the only integral theory which can be applied to materials which deform plastically. Only its range of applicability need be determined.

In the development of our formalism which was applied to the mechanical response of reactor grade graphite no basic assumption, aside from rate independence, was made concerning the nature of the material up to the point the form of the kernel functions were assumed. Because of this we may reasonably expect that this theory will be applicable to other rate independent materials and perhaps to even materials with almost rate independent response. In particular this formulation seems readily applicable to the description of soils, concrete, and strain hardening metals.

Looking retrospectively at the mathematical model derived here we see that the equations look similar to those equations encountered in the linear theory of viscoelasticity. From this observation we may term our theory a theory of viscoplasticity, or some other term which may prove more appropriate.

The question arises as to the use of our theory in solving boundary value problems. It would be very desirable to solve even the most simple boundary value problem. There are two basic stumbling blocks to the solution of this type of problem. First, we must discover how to convert the natural boundary conditions, given in terms of time and position, to boundary conditions in terms of the arc length. Second, we must find a solution to the field equations which become unwieldy when written in terms of the nonlinear arc length.

It should be mentioned that our theory is capable of predicting a yield surface. In fact, by following the work of Morgan [53] a yield surface can be predicted which can be interpreted as the usual yield surface encountered in the classical plasticity theories. It is possible that by this means one equation can be developed which will descirbe the entire range of an elastic-plastic material. That is, it may be possible to develop one equation which can describe the cyclic straining behavior of an elastic-plastic material in both the elastic and plastic ranges.

Returning to graphite it would be very instructive to obtain both one dimensional and three dimensional hysteresis data for the Bauschinger effect in reactor grade graphite. Because of a lack of data for graphite in reversed tension and compression the case of hysteresis around the origin of the stress-strain curve was not analyzed. It is felt that the present theory will be capable of predicting this type of behavior adequately.

There are areas, outside of mechanics, where the type of analysis we have employed might prove extremely useful. Pipkin and Rivlin have shown that a theory of the type we have developed is directly applicable to the theory of magnetism [54]. Our theory might also be applied to some of the problems in the biological sciences where input-output systems can represent the biological process under consideration.

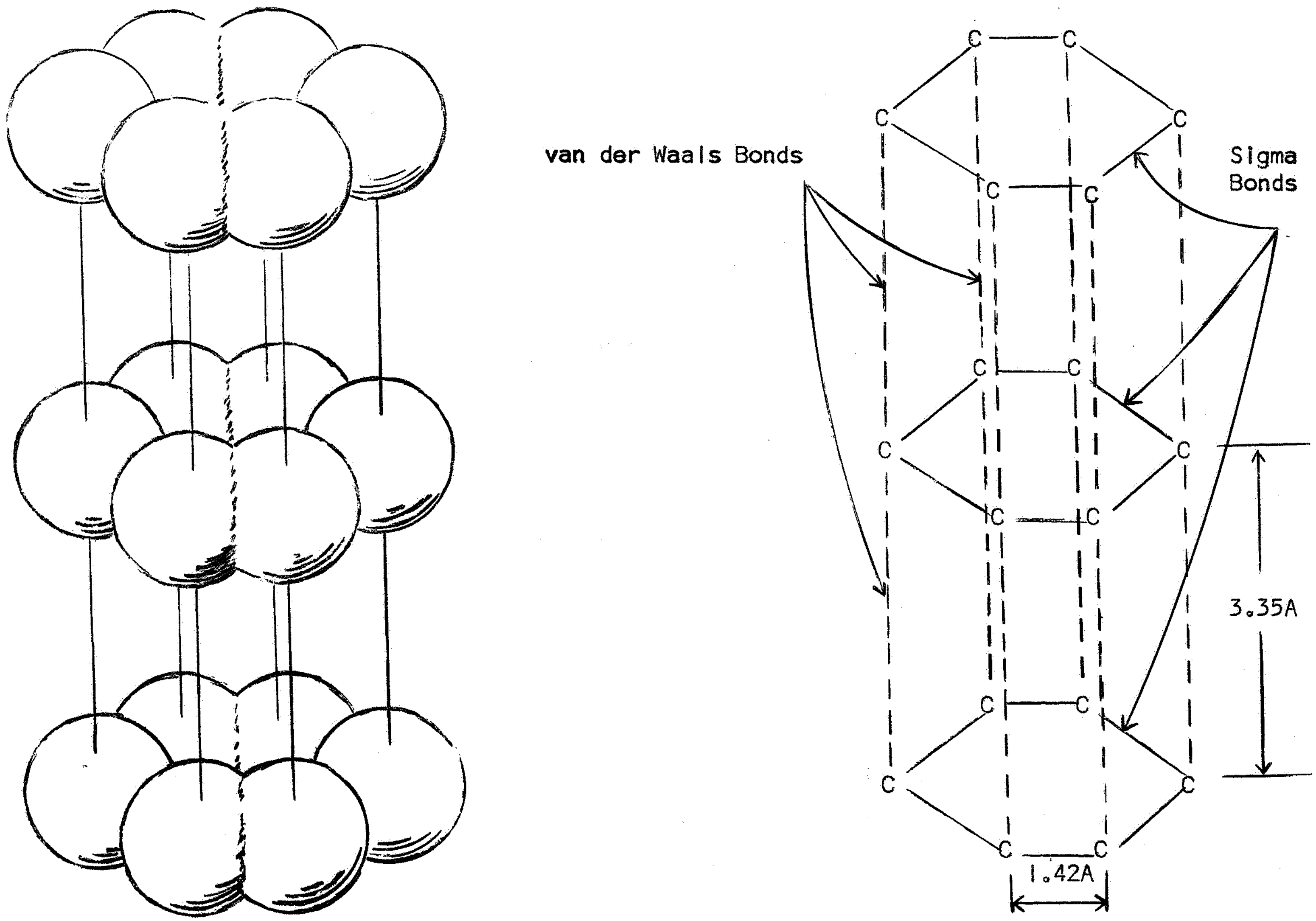

Fig. 1. Graphite Crystal Structure

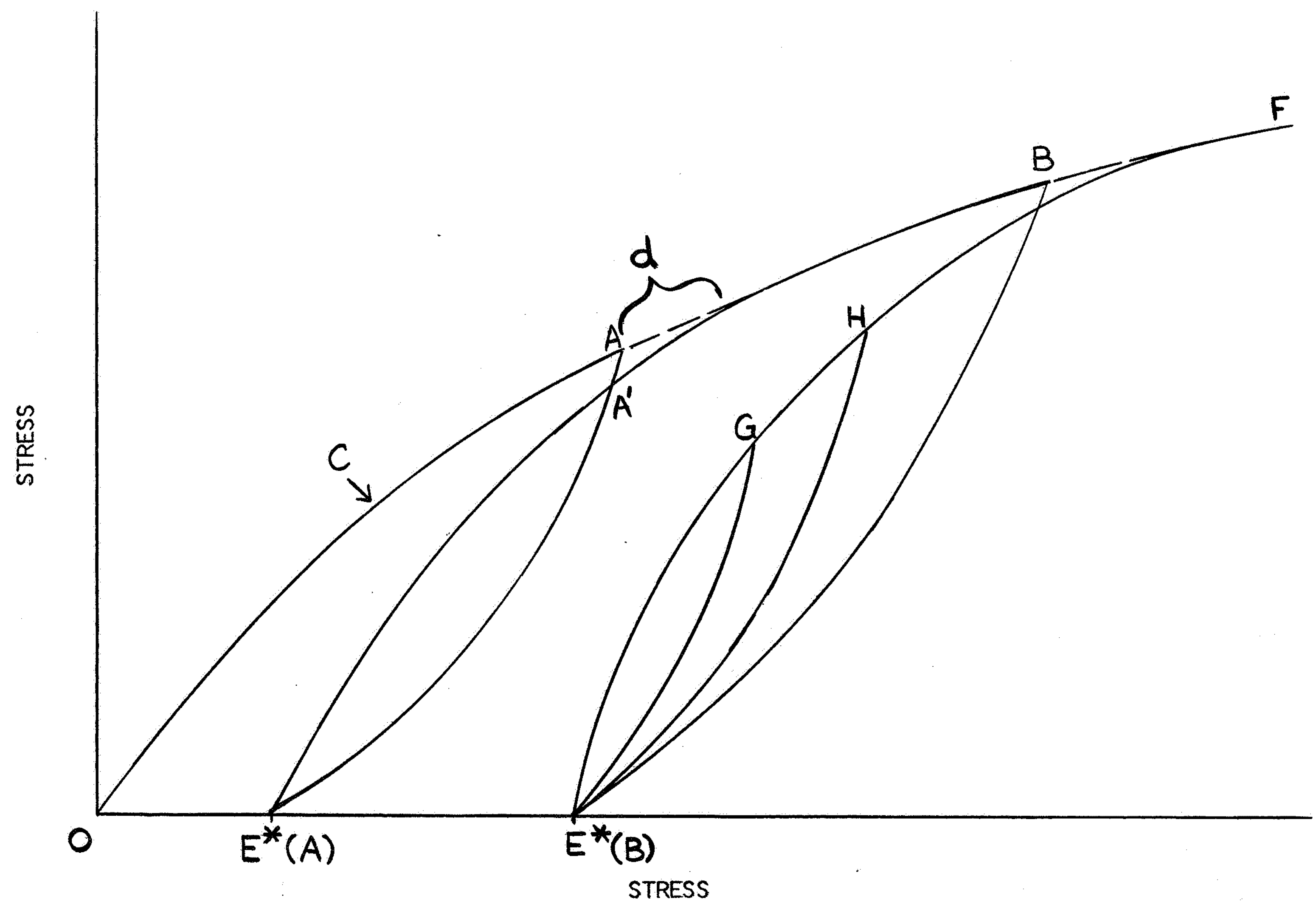

Fig. 2. Graphite Stress-strain Curve

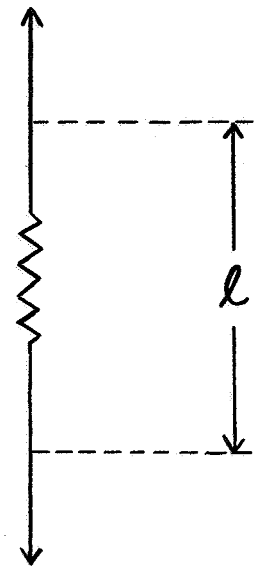

Fig. 3. Spring Element

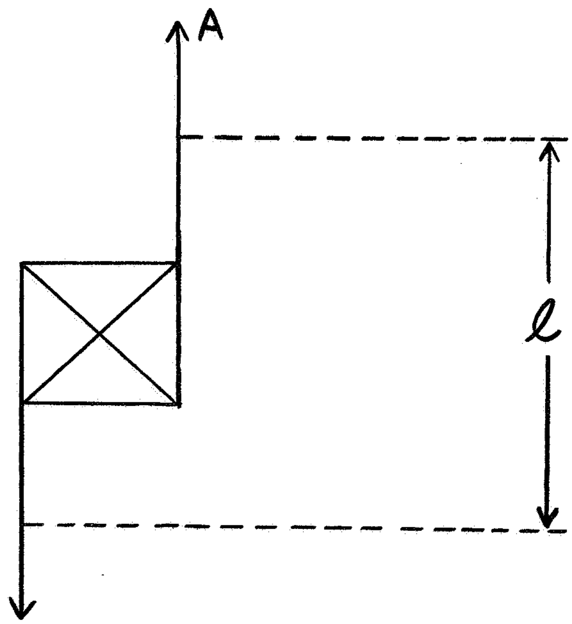

Fig. 4. Friction Block Element

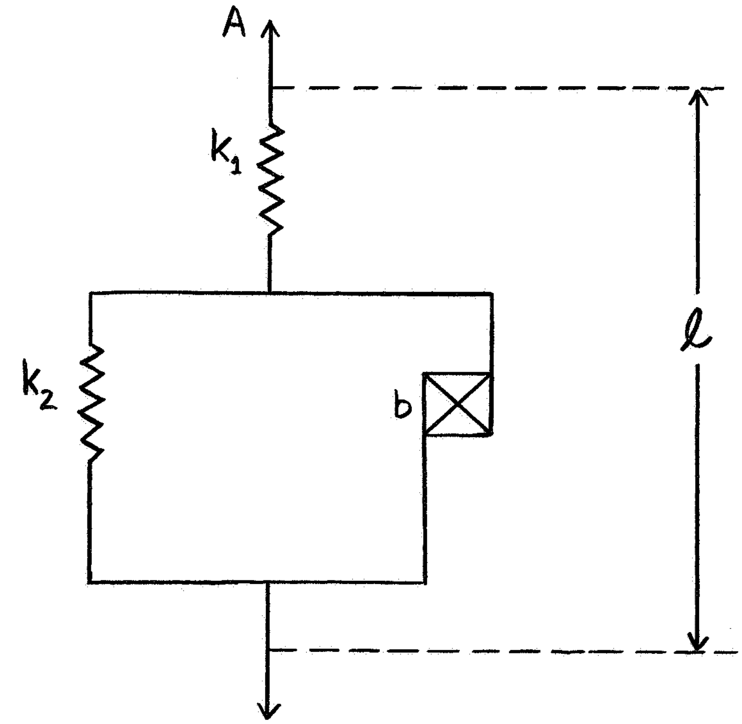

Fig. 5. Elastic-plastic Model

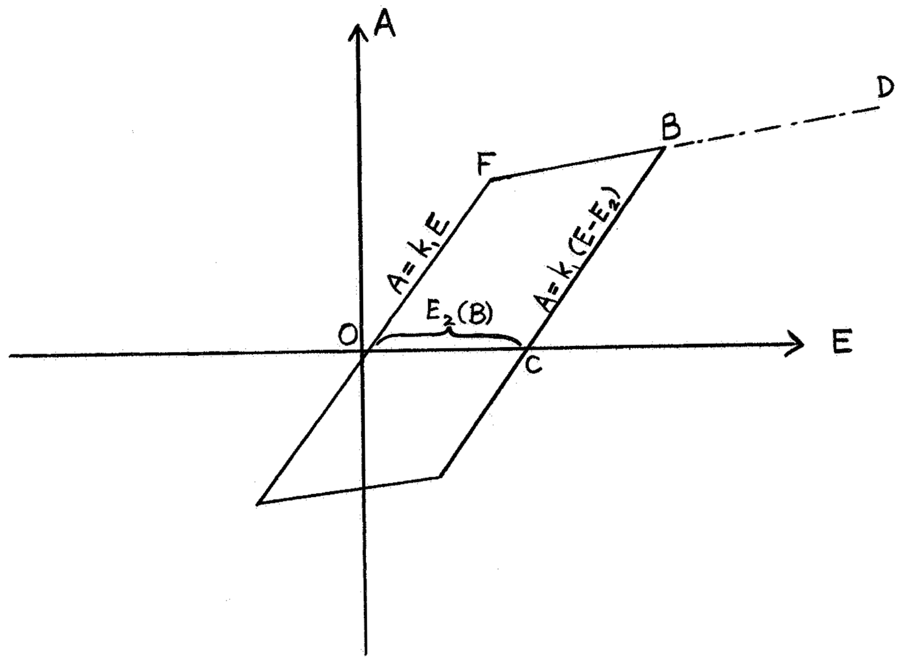

Fig. 6. Elastic-plastic stress-strain curve

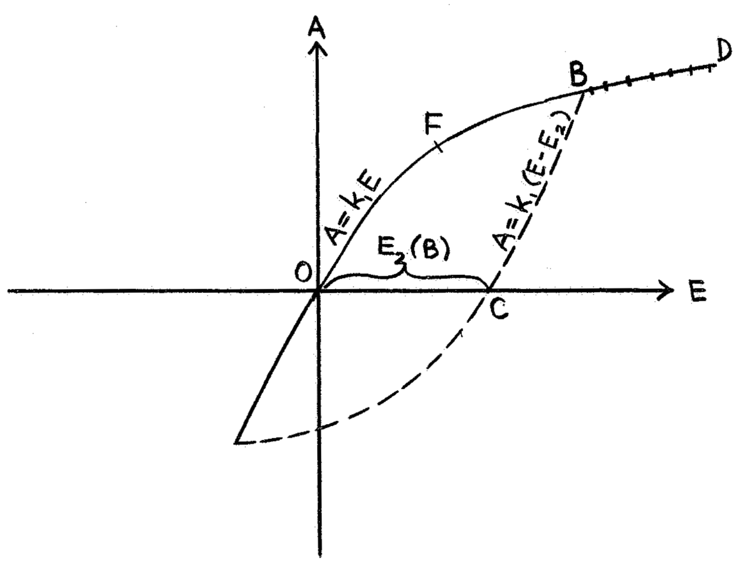

Fig. 7. Generalized Elastic-plastic curve

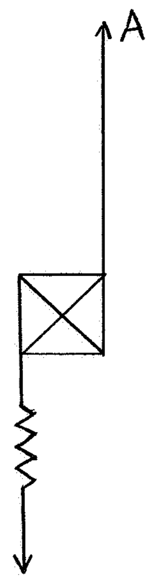

Fig. 8. Basic Graphite Element

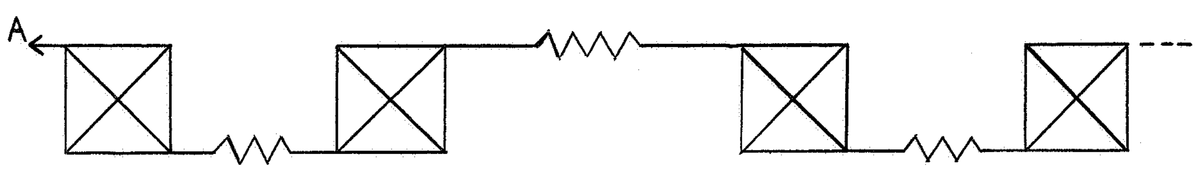

Fig. 9. One Dimensional Graphite Model

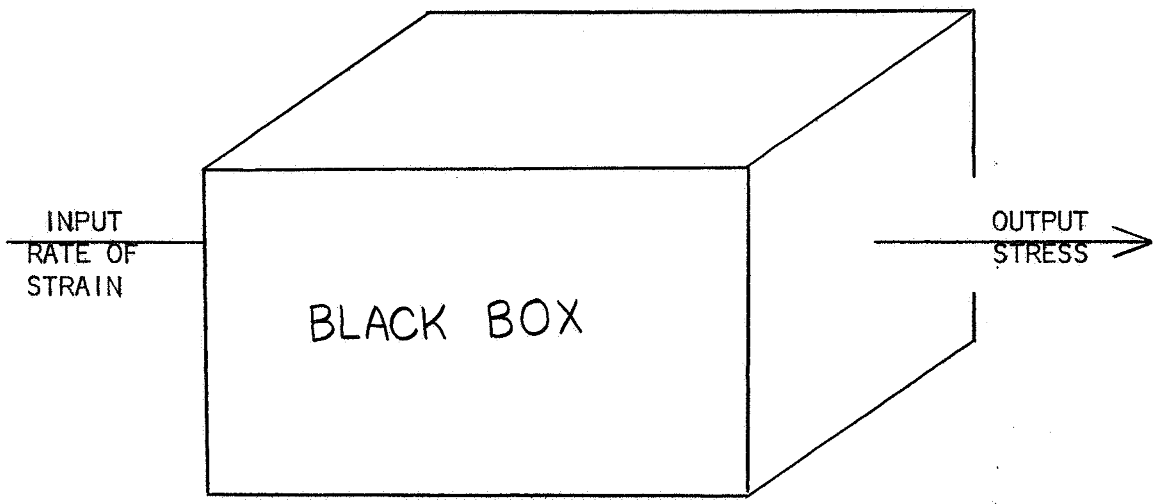

Fig. 10. Black Box

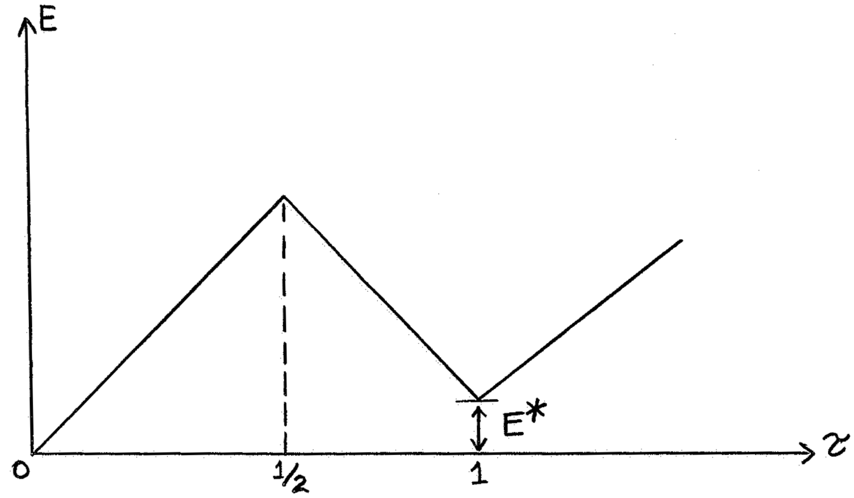

Fig. IIa. The Straining Program

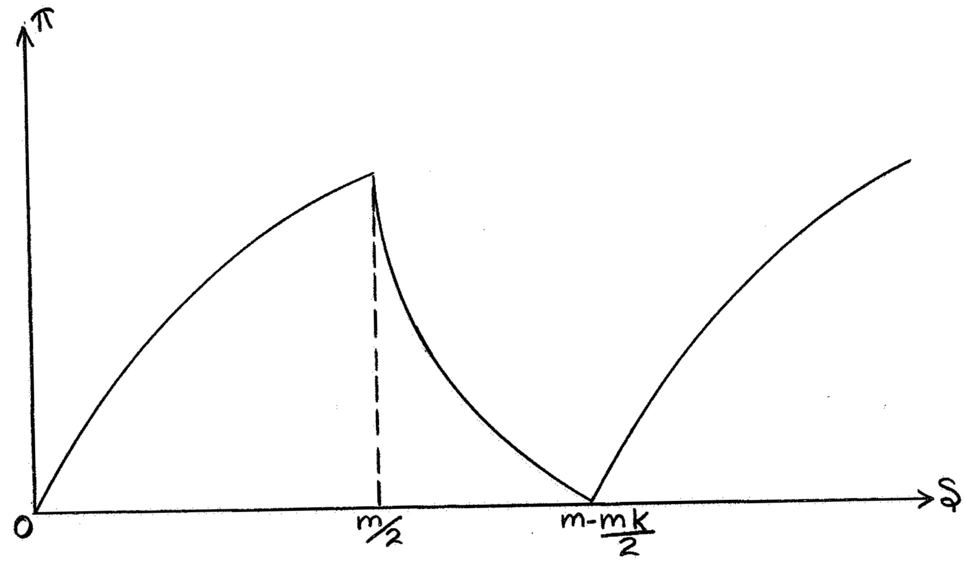

Fig. IIb. The Corresponding Stress vs. Arc Length Plot

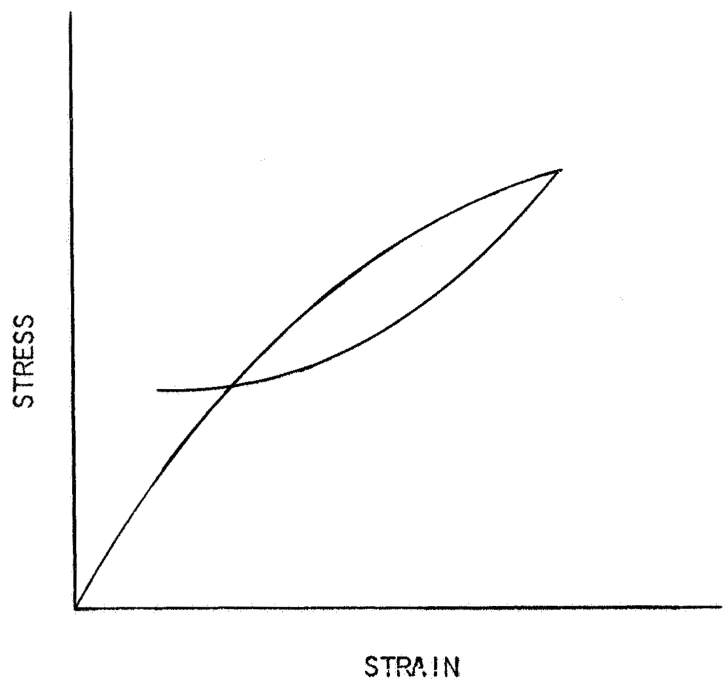

Fig. 12. Stress-Strain Curve for Equation (4.25)

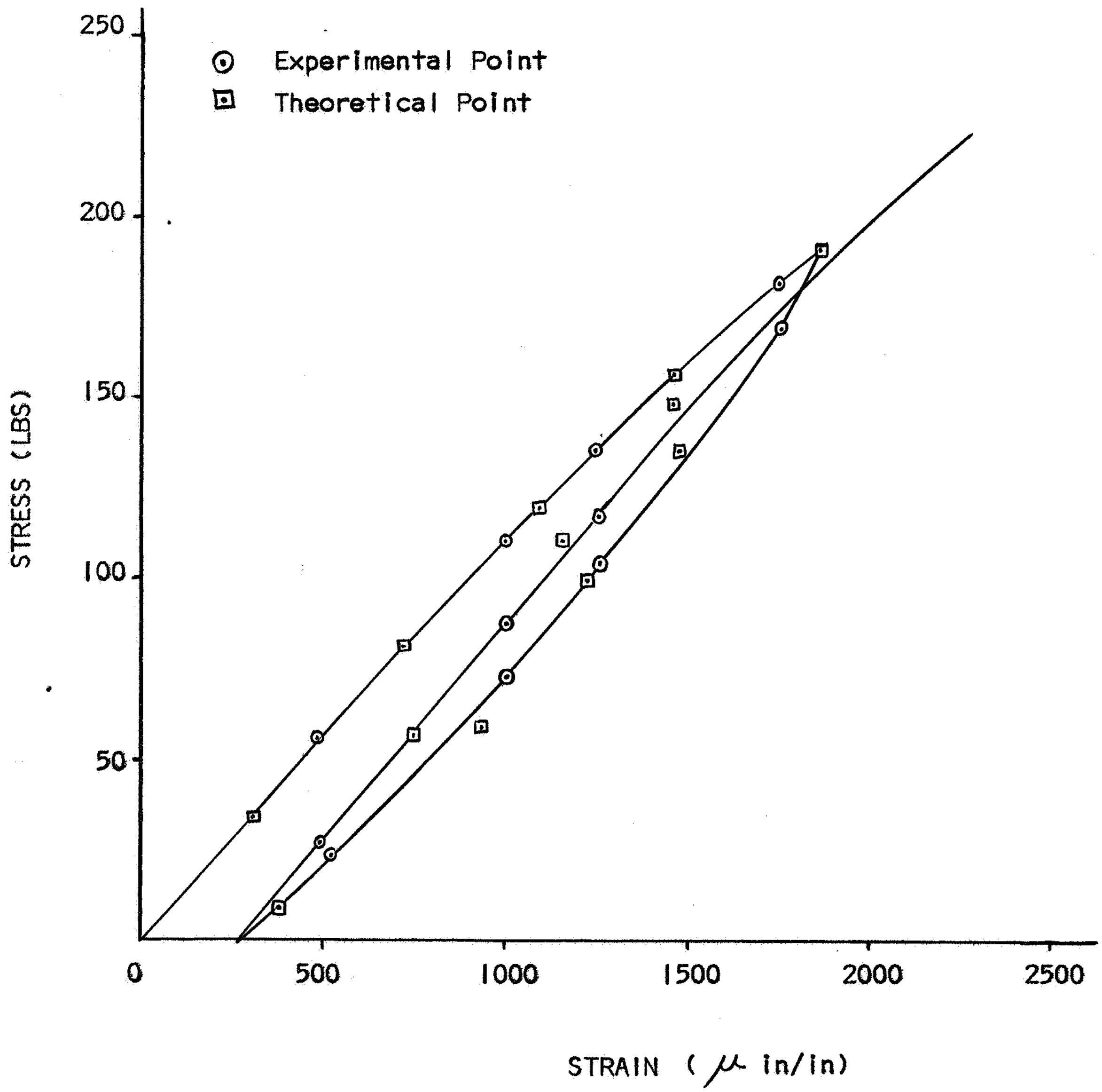

Fig. 13. Comparison of Theoretical and Experimental Stress-Strain Curves

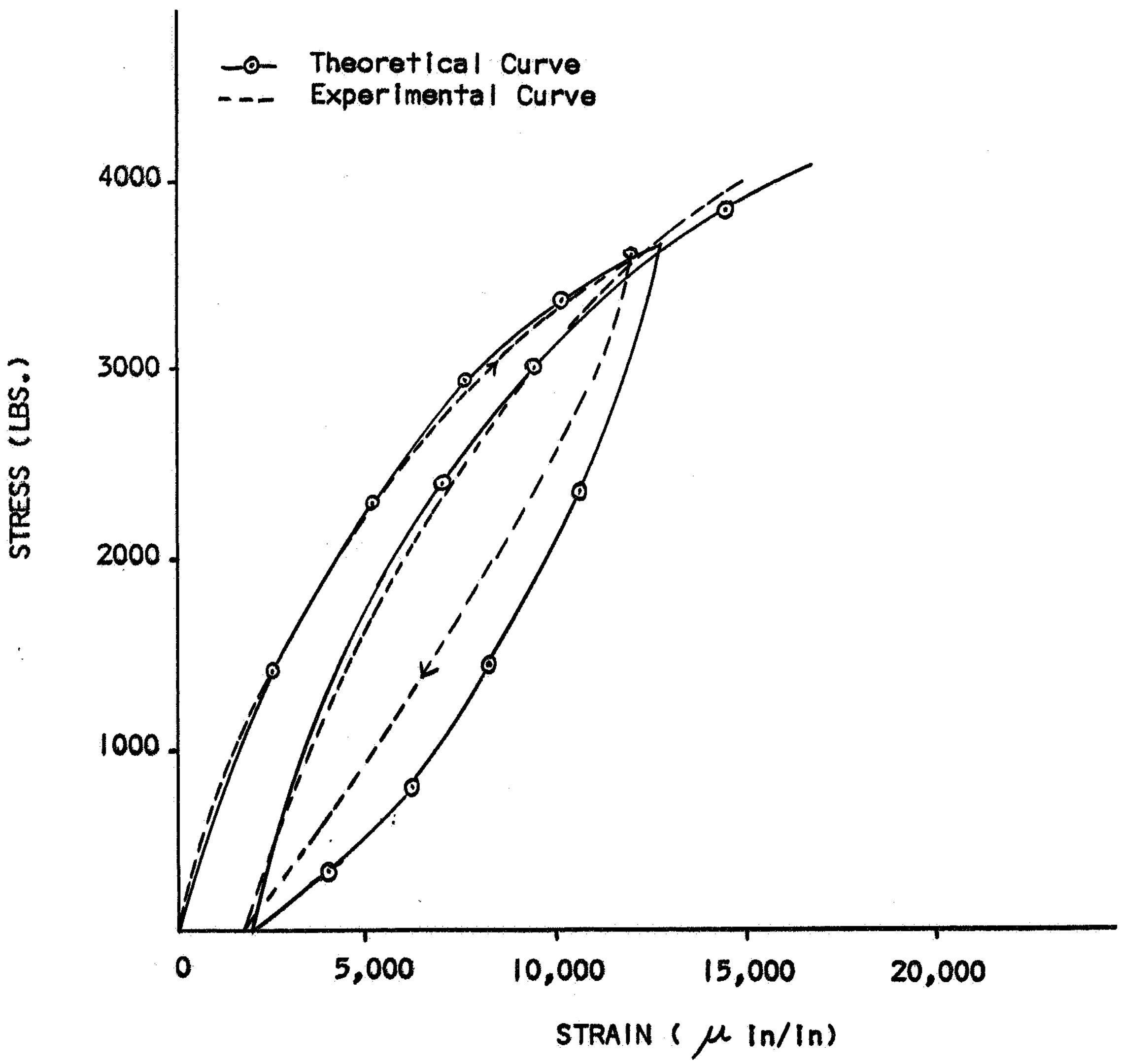

Fig. 14a. Comparison of Theory and Experiment for the 22-direction (Case I). Graphite specimen number ATJ-I-B-2-I compressed parallel to the 22-direction.

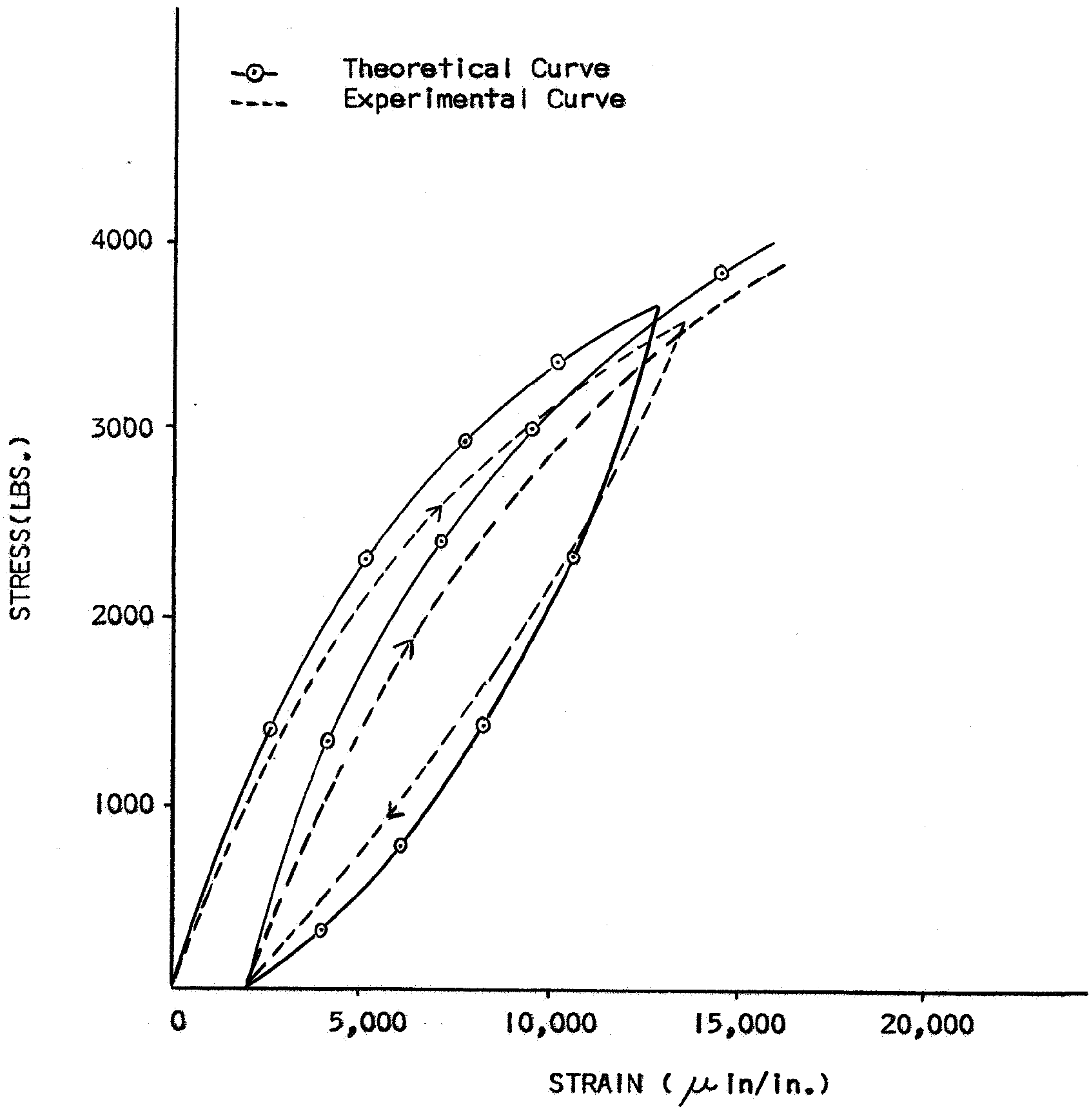

Fig. 14b. Comparison of Theory and Experiment for the 33-direction (Case I). Graphite specimen number ATJ-I-C-2-I compressed parallel to the 33-direction.

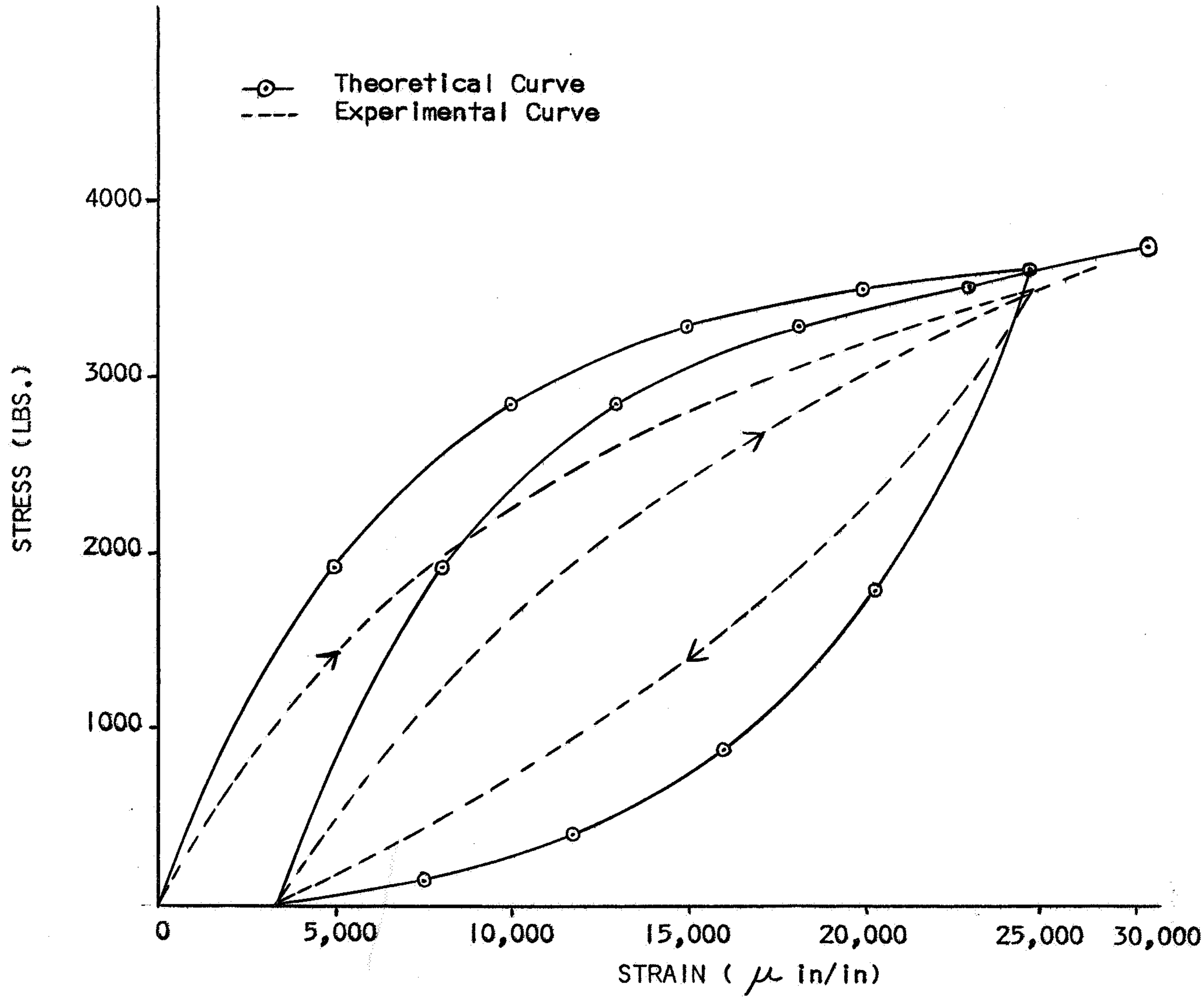

Fig. 15. Comparison of Theory and Experiment (Case II). Graphite specimen number ATJ-I-G-2-I compressed parallel to the II-direction

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